1 Super-symmetric sigma-model and Landau-Ginzburg model

- Lagrangian of the super-symmetric non-linear sigma model with the Kahler target space X is

$$\mathcal{L} = -g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\phi^{\bar{j}} + ig_{i\bar{j}}\overline{\psi}_-^{\bar{j}}(D_0+D_1)\psi_-^i + ig_{i\bar{j}}\overline{\psi}_+^{\bar{j}}(D_0-D_1)\psi_+^i + R_{i\bar{j}k\bar{l}}\psi_+^i\psi_-^j\psi_-^{\bar{j}}\psi_+^{\bar{l}}. \quad (1)$$

where the target space Kahler metric is

$$g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K. \quad (2)$$

Field ϕ defines mappings $\phi : \Sigma \to X$, while fermionic fields are sections of the bull-back of tangent bundle

$$\psi_\pm \in \Gamma(\Sigma,\, \phi^*TM^{(1,0)} \otimes S_\pm), \quad \bar{\psi}_\pm \in \Gamma(\Sigma,\, \phi^*TM^{(1,0)} \otimes S_\pm) \quad (3)$$

so the covariant derivatives are

$$D_\mu\psi_\pm^i = \partial_\mu\psi_\pm^i + \partial_\mu\phi^j\Gamma^i_{jk}\psi_\pm^k. \quad (4)$$

- The action can be compactly written using single D-term by

$$\mathcal{L} = \int d^4\theta\; K(\Phi,\overline{\Phi}). \quad (5)$$

- One can deform the model to Landau-Ginzburg model, including super-potential by F-term

$$\delta\mathcal{L} = -\frac{1}{4}g^{\bar{i}j}\partial_{\bar{i}}W\partial_jW - \frac{1}{2}D_i\partial_jW\psi_+^i\psi_-^j - \frac{1}{2}D_{\bar{i}}\partial_{\bar{j}}\overline{W\psi}_-^{\bar{i}}\overline{\psi}_+^{\bar{j}} \quad (6)$$

which is written using super-fields by

$$\delta\mathcal{L} = \frac{1}{2}\left(\int d^2\theta\; W(\phi) + h.c.\right) \quad (7)$$

- One can add topological term into path integral preserving super-symmetry using some $B \in H^2(M,\mathbb{R})$:

$$\exp\left(i\int_\Sigma \phi^*B\right). \quad (8)$$

- The target space is 'coupling constant' of the sigma-model which is to be renormalized. In the one-loop approximation (doesn't affected by fermions) RG equation is

$$\mu\frac{dg_{IJ}(\mu)}{d\mu} = \frac{1}{2\pi}R_{IJ}(g(\mu)), \quad g_{IJ}(\Lambda_{UV}) = g_{IJ}(\mu) + \frac{1}{2\pi}\log\left(\frac{\Lambda_{UV}}{\mu}\right)R_{IJ} \quad (9)$$

- There are three cases:
  - $R_{IJ} > 0$ - asymptotic freedom. The bare metric is large (if we fix physical metric) when we make Λ_{UV} large, so there perturbation theory works well, theory is free. On the opposite - in the IR the model is strongly coupled.
  - $R_{IJ} = 0$ - the theory is conformal. The CY case is 'close' to be so. For LG models - one also have to demand quasi-homogeneity of W, as it flows 'trivially' (see below). Hoped to flow to IR CFT.
  - $R_{IJ} < 0$ theory is ill-defined case.

  This result is not exact (non-trivial four loops for CY, exact for Grassmanians and Hermitean spaces).

- However the renormalization of the Kahler class can be obtained from anomalies, and given by

$$[\omega](\mu) \to [\omega](\Lambda) + \log(\mu/\Lambda)c_1(M) \quad (10)$$

where $[\omega] \in H^2(M,\mathbb{R})$ is complexified, includes B-field $B \in H^2(M,\mathbb{R})/H^2(M,\mathbb{Z})$. Scale + axial anomaly absorb one parameter.

- We will be computing anomalies of the theory using Groethendieck-Riemann-Roch formula

$$\chi(E) = \sum_k(-1)^k\dim H^k(E) = \int_\Sigma \mathrm{ch}(E)\mathrm{td}(T\Sigma), \quad \mathrm{td}(F) = \det\left(\frac{F}{1-e^{-F}}\right), \quad \mathrm{ch}(F) = \det(1+F) \quad (11)$$

For our case

$$\Sigma = 1 + \frac{1}{2}c_1(\Sigma) \quad (12)$$

2 Super-symmetry

- The symmetry algebra of the theory is $\mathcal{N} = (2,2)$ super-Poincare in 2d:

$$[P^{\pm}, M] = \pm M, \quad \{\mathcal{Q}_{\pm}, \overline{\mathcal{Q}}_{\pm}\} = -2i\partial_{\pm} = -2P_{\pm}, \quad [M, \mathcal{Q}_{\pm}] = \pm i\mathcal{Q}_{\pm}, \quad [M, \overline{\mathcal{Q}}_{\pm}] = \pm i\overline{\mathcal{Q}}_{\pm}, \quad \mathcal{Q}_{\pm}^{\dagger} = \overline{\mathcal{Q}}_{\pm} \tag{13}$$

which can be represented in super-space by

$$\mathcal{Q}_{\pm} = \frac{\partial}{\partial\theta^{\pm}} + i\bar{\theta}^{\pm}\partial_{\pm}, \quad \overline{\mathcal{Q}}_{\pm} = \frac{\partial}{\partial\bar{\theta}^{\pm}} + i\theta^{\pm}\partial_{\pm}, \quad P^{\pm} = \frac{\partial}{\partial x^{\pm}}, \quad M = x^{+}\frac{\partial}{\partial x^{+}} - x^{-}\frac{\partial}{\partial x^{-}} \tag{14}$$

$$x^{\pm} = x^0 \pm x^1, \quad (\theta^{\pm})^* = \bar{\theta}^{\pm} \tag{15}$$

Super-covariant derivatives, commuting with Poincare algebra

$$\{D_{\pm}, \overline{D}_{\pm}\} = 2i\partial_{\pm}, \quad \{D_{\pm}, \mathcal{Q}_{\pm}\} = \{D_{\pm}, \overline{\mathcal{Q}}_{\pm}\} = 0, \quad \{\overline{D}_{\pm}, \mathcal{Q}_{\pm}\} = \{\overline{D}_{\pm}, \overline{\mathcal{Q}}_{\pm}\} = 0, \tag{16}$$

can be defined by

$$D_{\pm} = \frac{\partial}{\partial\theta^{\pm}} - i\bar{\theta}^{\pm}\partial_{\pm}, \quad \overline{D}_{\pm} = -\frac{\partial}{\partial\bar{\theta}^{\pm}} + i\theta^{\pm}\partial_{\pm} \tag{17}$$

- Chiral super-field is defined by constraints

$$\overline{D}_{\pm}\Phi = 0 \tag{18}$$

which can be solved by

$$\Phi(x, \theta, \bar{\theta}) = \phi(y^{\pm}) + \theta^{\alpha}\psi_{\alpha}(y^{\pm}) + \theta^{+}\theta^{-}F(y^{\pm}), \quad y^{\pm} = x^{\pm} - i\theta^{\pm}\bar{\theta}^{\pm} \tag{19}$$

Twisted chiral super-field is defined by constraints

$$\overline{D}_{+}U = 0, \quad D_{-}U = 0 \tag{20}$$

which can be solved by

$$U(x, \theta, \bar{\theta}) = v(\tilde{y}^{\pm}) + \theta^{+}\bar{\chi}_{+}(\tilde{y}^{\pm}) + \bar{\theta}^{-}\chi_{-}(\tilde{y}^{\pm}) + \theta^{+}\bar{\theta}^{-}E(\tilde{y}^{\pm}), \quad \tilde{y}^{\pm} = x^{\pm} \mp i\theta^{\pm}\bar{\theta}^{\pm} \tag{21}$$

writing generator of super-symmetry as

$$\delta = \epsilon_{+}\mathcal{Q}_{-} - \epsilon_{-}\mathcal{Q}_{-} - \bar{\epsilon}_{+}\overline{\mathcal{Q}}_{-} + \bar{\epsilon}_{-}\overline{\mathcal{Q}}_{+}, \quad \hat{\delta}^{\dagger} = -\hat{\delta} \tag{22}$$

one finds transformation rules in components

$$\begin{array}{ll} \delta\phi = \epsilon_{+}\psi_{-} - \epsilon_{-}\psi_{+} & \delta v = \bar{\epsilon}_{+}\chi_{-} - \epsilon_{-}\bar{\chi}_{+} \\ \delta\psi_{+} = +2i\epsilon_{-}\partial_{+}\phi + \epsilon_{+}F & \delta\bar{\chi}_{+} = 2i\bar{\epsilon}_{-}\partial_{+}v + \bar{\epsilon}_{+}E \\ \delta\psi_{-} = -2i\epsilon_{+}\partial_{-}\phi + \epsilon_{-}F & \delta\chi_{-} = -2i\epsilon_{+}\partial_{-}v + \epsilon_{-}E \\ \delta F = -2i\bar{\epsilon}_{+}\partial_{-}2i\bar{\epsilon}_{-}\partial_{+}\psi_{-} & \delta E = -2i\epsilon_{+}\partial_{-}\bar{\chi}_{+} - 2i\bar{\epsilon}_{-}\partial_{+}\chi_{-} \end{array} \tag{23}$$

The lowest components satisfy constraints:

$$[\overline{\mathcal{Q}}_{\pm}, \phi] = 0, \quad [\overline{\mathcal{Q}}_{+}, v] = [\mathcal{Q}_{-}, v] = 0 \tag{24}$$

Conversely, if one got such fields, the whole multiplet can be constructed by the action of rising operators.

- Real vector multiplet V is real scalar super-field, modulo transformations

$$V \to V + i(\bar{A} - A) \tag{25}$$

Fixing it properly, one can find

$$V = \theta^{-}\bar{\theta}^{-}(v_0 - v_1) + \theta^{+}\bar{\theta}^{+}(v_0 + v_1) - \theta^{-}\bar{\theta}^{+}\sigma - \theta^{+}\bar{\theta}^{-}\bar{\sigma} + \tag{26}$$

$$+ i\theta^{-}\theta^{+}(\bar{\theta}^{-}\bar{\lambda}_{-} + \bar{\theta}^{+}\bar{\lambda}_{+}) + i\bar{\theta}^{+}\bar{\theta}^{-}(\theta^{-}\lambda_{-} + \theta^{+}\lambda_{+}) + \theta^{-}\theta^{+}\bar{\theta}^{+}\bar{\theta}^{-}D$$

The field-strength is coming from the twisted chiral super-field

$$\Sigma = \overline{D}_{+}D_{-}V, \quad \overline{D}_{+}\Sigma = 0, \quad D_{-}\Sigma = 0 \tag{27}$$

which decomposes as

$$\Sigma = \sigma(\tilde{y}) + i\theta^{+}\bar{\lambda}_{+}(\tilde{y}) - i\bar{\theta}^{-}\lambda_{-}(\tilde{y}) + \theta^{+}\bar{\theta}^{-}[D(\tilde{y}) - iv_{01}(\tilde{y})], \quad v_{01} = \partial_0 v_1 - \partial_1 v_0 \tag{28}$$

- Non-renormalization theorem:
  - F term and twisted F term doesn't mix under renormalization. D term doesn't affect renormalization of F and twisted F terms (by promoting couplings in D to fields).
  - Super-potential doesn't renormalize (by demoting fields to couplings).

  However if we write effective action, and integrate some fields out - the couplings can come.

- There are two R-symmetries

$$(e^{i\alpha F_V}\mathcal{F})(x,\theta^{\pm},\bar{\theta}^{\pm}) = e^{i\alpha q_V}\mathcal{F}(x, e^{-i\alpha}\theta^{\pm}, e^{i\alpha}\bar{\theta}^{\pm}) \tag{29}$$

$$(e^{i\alpha F_A}\mathcal{F})(x,\theta^{\pm},\bar{\theta}^{\pm}) = e^{i\alpha q_A}\mathcal{F}(x, e^{\mp i\alpha}\theta^{\pm}, e^{\pm i\alpha}\bar{\theta}^{\pm}) \tag{30}$$

  Note that super-symmetry change charges by

$$[F_A, \mathcal{Q}_{\pm}] = \mp\mathcal{Q}_{\pm}, \quad [F_A, \overline{\mathcal{Q}}_{\pm}] = \pm\overline{\mathcal{Q}}_{\pm}, \quad [F_V, \mathcal{Q}_{\pm}] = -\mathcal{Q}_{\pm}, \quad [F_V, \overline{\mathcal{Q}}_{\pm}] = \overline{\mathcal{Q}}_{\pm} \tag{31}$$

  The axial R-charge of chiral super multiplet is $q_A = 0$ so the components transform as

$$F_A: \ \phi \mapsto \phi, \quad \psi_{\pm} \mapsto e^{\mp i\alpha}\psi_{\pm}, \quad F \mapsto F \tag{32}$$

  The vector symmetry is broken by superpotential, unless it is homogeneous of degree k wrt the Φ. In this case $q_V = 2/k$ and components transform by

$$F_V: \ \phi \mapsto e^{2i\alpha/k}\phi, \quad \psi_{\pm} \mapsto e^{(2/k-1)i\alpha}\psi_{\pm}, \quad F \mapsto e^{(2/k-2)i\alpha}F \tag{33}$$

  Twisted field can be obtained by replacement $\theta^- \leftrightarrow \bar{\theta}^-$ which replace $F_V \leftrightarrow F_A$ so $q_V = 0$ and the vector transformation rules are

$$F_V: \ v \mapsto v, \quad \chi^- \mapsto e^{i\alpha}\chi^-, \quad \bar{\chi}^+ \mapsto e^{-i\alpha}\bar{\chi}^+, \quad E \mapsto E \tag{34}$$

  Again, if twisted super-potential is homogenious of degree k than

$$F_A: \ v \mapsto e^{2i\alpha/k}v, \quad \chi^- \mapsto e^{(2/k-1)i\alpha}\chi^-, \quad \bar{\chi}^+ \mapsto e^{(2/k-1)i\alpha}\bar{\chi}^+, \quad E \mapsto e^{(2/k-2)i\alpha}E \tag{35}$$

3 Target spaces

- The almost complex manifold X is the manifold equipped with smooth map J

$$J: TX \to TX \ \text{ s.t. } \ J^2 = -1, \quad J\left(\frac{\partial}{\partial x^a}\right) = J^b_a\frac{\partial}{\partial x^b} \tag{36}$$

  This splits

$$T_{\mathbb{C}}X = TX \oplus \overline{T}X: \ J|_{TX} = i, \quad J|_{\overline{T}X} = -i \tag{37}$$

  which induces decomposition on differential forms

$$\Omega^n(M) = \bigoplus_{p+q=n} \Lambda^p T^*M \otimes \Lambda^q \overline{T}^* M = \bigoplus_{p+q=n} \Omega^{p,q}(M) \tag{38}$$

  De-Rham derivative can be decomposed to

$$d = \partial + \bar{\partial} + ... \tag{39}$$

  where

$$\partial|_{\Omega^{p,q}} = P|_{p+1,q}\cdot d, \quad \bar{\partial}|_{\Omega^{p,q}} = P|_{p,q+1}\cdot d \tag{40}$$

  and $P|_{p,q}$ is the projector on $\Omega^{p,q}(M)$.

- If J satisfies $\bar{P}[PX, PY] = 0$ then J can be 'integrated', and one can find complex coordinates with holomorphic transition maps. Equivalently, one can demand that $\bar{\partial}^2 = 0$. In this case

$$d = \partial + \bar{\partial}; \quad d^2 = 0 \Rightarrow \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0 \tag{41}$$

  The deformations of complex structure $J \mapsto J + \epsilon$ satisfying $J^2 = 0$ modulo coordinate changes are elements of $H^1_{\bar{\partial}}(TM)$ i.e. such

$$\epsilon = \epsilon^a_{\bar{b}} d\bar{z}^{\bar{b}} \otimes \frac{\partial}{\partial z^a} \tag{42}$$

  that satisfy $\partial_{\bar{a}}\epsilon^c_{\bar{b}} - \partial_{\bar{b}}\epsilon^c_{\bar{a}} = 0$ modulo $\epsilon^c_{\bar{b}} = \partial_{\bar{b}}v^c$. One can also consider this as deformation of $\bar{\partial}$ operator.

- One can compute that Laplacians coincide

$$\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial} \tag{43}$$

and show that

$$\mathrm{Harm}^{r,s}_{\bar{\partial}} = H^{r,s}_{\bar{\partial}} \tag{44}$$

so by the Hodge theorem one can show that de-Rham cohomology are decomposable

$$H^n(M) = \bigoplus_{p+q=n} H^{p,q}_{\bar{\partial}}(M) \tag{45}$$

- The Hodge numbers $h^{p,q} = \dim\, H^{p,q}(M)$ got properties
  - $h^{p,q} = h^{n-p,n-q}$
  - $h^{p,q} = h^{q,p}$
  - For CY triviality of $H^1(X)$ implies $h^{0,2} = 0$.
- The metric is said to be Hermitian, and define positive-defined inner product

$$g : TM \otimes \overline{T}M \to \mathbb{C}, \quad g = g_{i\bar{j}} dz^i \otimes dz^{\bar{j}}. \tag{46}$$

if is satisfy

$$g(X,Y) = g(JX, JY) \quad \Leftrightarrow \quad J_{ab} = -J_{ba}, \quad J_{ab} = J^c_a g_{cb}. \tag{47}$$

One can define associated $(1,1)$-form by

$$\omega(X,Y) = g(JX, Y), \quad \omega = i g_{i\bar{j}} dz^i \wedge dz^{\bar{j}} \tag{48}$$

- The manifold is said to be Kahler and the form defines Kahler class $[\omega] \in H^{1,1}(X)$ if

$$d\omega = 0, \quad \partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}} \tag{49}$$

Locally one can find such Kahler potential K, that $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$.

- On Kahler manifold

$$\Gamma^i_{j\bar{k}} = ... = \Gamma^{\bar{i}}_{\bar{j}k} = 0, \quad \Gamma^i_{jk} = g^{i\bar{l}} \partial_j g_{\bar{l}k}, \quad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = g^{\bar{i}l} \partial_{\bar{j}} g_{l\bar{k}} \tag{50}$$

- The property of being Kahler survives rescaling of the metric. Kahler cone is the cone in $H^{1,1}(M)$ of such ω, for which associated metrics define strictly positive volume for any sub-manifold. Inside Kahler cone one can slightly deform ω by the any element of $H^{1,1}(M)$, so these are deformations of Kahler structure.
- The Calabi-Yau manifold is such Kahler, that $c_1(TM) = [\frac{1}{2\pi}\mathcal{R}] = 0$. For simply connected, condition $\Omega^{d,0}(M) = \mathcal{O}$ (i.e. having non-vanishing highest holomorphic form) is equivalent. The equivalence is by taking determinant bundle.
- In the case of CY-3 one can use highest holomorphic form to show

$$H^1_{\bar{\partial}}(TX) = H^{2,1}_{\bar{\partial}}(X) \tag{51}$$

so $h^{2,1}$ is the number of complex deformations, $h^{1,1}$ is the number of Kahler deformations.

- CY theorem: For complex manifold fixed complex structure and $c_1(X) = 0$ there exists unique (up to scaling) Kahler metric which
  - Has fixed $[\Omega] \in H^{1,1}(X)$ (i.e. periods).
  - The curvature is zero $\mathcal{R} = 0$

4 Non-linear from linear sigma models

- The gauged linear sigma-model is

$$\mathcal{L} = \int d^4\theta \left(\sum_{i=1}^{N} \overline{\Phi}_i e^{Q_{ia}V_a} \Phi_i - \sum_{a=1}^{k} \frac{1}{2e_a^2} \overline{\Sigma}_a \Sigma_a \right) + \frac{1}{2} \left(\int d^2\tilde{\theta} \sum_{a=1}^{k} (-t_a \Sigma_a) + c.c. \right) \tag{52}$$

where we used real vector-multiplet. The parameters $t_a = r_a - i\theta_a$ are called FI parameters. The most interesting pieces are potential term

$$U = \sum_{i=1}^{N} |Q_{ia}\sigma_a|^2 |\phi_i|^2 + \sum_{a=1}^{k} \frac{(e^a)^2}{2} (Q_{ia}|\phi_i|^2 - r_a)^2 \tag{53}$$

and theta-terms

$$\mathcal{L}_\theta = \sum_a \theta_a (v_a)_{01} \tag{54}$$

- We are interested in phase where $\phi \neq 0,\ \sigma = 0$. In the strong coupling regime $e_a^2 \to +\infty$:
  - ϕ_i tangent to vacuum manifold are massless.
  - ϕ_i transverse to vacuum manifold have mass $e\sqrt{2r}$
  - v_μ eat one mass-less scalar and get mass $e\sqrt{2r}$
  - The only massless fermions are those, which are tangent to vacuum

$$\sum_{i=1}^{N} \overline{\phi}_i \psi_\pm^i = 0, \quad \sum_{i=1}^{N} \phi_i \overline{\psi}_\pm^i = 0 \tag{55}$$

- Using the equations of motion for vector and auxiliary fields, with kinetic term omitted, one can find that

$$\int_{C \in H_2(X)} (\omega - iB) = \frac{1}{2\pi} t_C \tag{56}$$

- Large class of examples can, for which the target is defined by Hamiltonian reduction (wrt the $\omega = \sum_k d\phi_k \wedge d\bar{\phi}_k$), can be defined so. The simplest:
  - Projective space $\mathbb{C}P^{N-1}$

$$\mathbb{C}P^{N-1} = \{(\phi_1, ..., \phi_N) \subset \mathbb{C}^N :\ H = |\phi_1|^2 + .. + |\phi_N|^2 - r = 0\} / \{U(1) :\ \phi_k \mapsto \phi_k e^{i\alpha}\} \tag{57}$$

  The gauge field constraint give

$$v_\mu = \frac{i}{2} \frac{\sum_{i=1}^{N} \left(\overline{\phi}_i \partial_\mu \phi_i - \partial_\mu \overline{\phi} \phi_i \right)}{\sum_{j=1}^{N} |\phi_j|^2} \tag{58}$$

  which defines metric for mass-less fields to be

$$ds^2 = \frac{r}{2\pi} g^{FS}. \tag{59}$$

  The gauge field also define connection, whose curvature is $c_1(\mathcal{O}(1)) = \frac{1}{2\pi}\omega^{FS}$ which generates $H^2(\mathbb{C}P^{N-1}, \mathbb{Z})$. Thus the B-field is

$$B = \frac{\theta}{2\pi} \omega^{FS} \tag{60}$$

  so, collecting together

$$[\omega] - i[B] = \frac{t}{2\pi} [\omega^{FS}] \tag{61}$$

  - Resolved conifold

$$\{(\phi_1, \phi_2, \phi_3, \phi_4) \subset \mathbb{C}^4 :\ H = |\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 - r = 0\} / U(1) \tag{62}$$

$$U(1) :\ (\phi_1, \phi_2, \phi_3, \phi_4) \mapsto (\phi_1 e^{i\alpha}, \phi_2 e^{i\alpha}, \phi_3 e^{-i\alpha}, \phi_4 e^{-i\alpha}) \tag{63}$$

  the open cell is

$$xz = yw,\ \text{where}\ x = \phi_1\phi_3,\ y = \phi_1\phi_4,\ z = \phi_2\phi_3,\ w = \phi_2\phi_4 \tag{64}$$

  The exceptional divisor is at $\phi_3 = \phi_4 = 0$ if $r > 0$ and defined by

$$\{(\phi_1, \phi_2) : |\phi_1|^2 + |\phi_2|^2 - r = 0\} / \{U(1) :\ (\phi_1, \phi_2) \mapsto (e^{i\alpha}\phi_1, e^{i\alpha}\phi_2)\} \tag{65}$$

- The geometric transition from resolved conifold to T^*S^3 is by considering it as a degenerate equation.

5 Topological twists

- The pair of combinations of SUSY generators

$$Q_A = \overline{Q}_+ + Q_-, \quad Q_B = \overline{Q}_+ + \overline{Q}_- \tag{66}$$

are scalar wrt the twister rotations

$$M_A = M + F_V, \quad M_B = M + F_A \tag{67}$$

This allows to define them on the curved Rhiman surface, as one don't need a covariantly constant spinor in order to have vanishing SUSY variation

$$\delta S = \int\limits_{\Sigma} \left(\nabla_\mu \epsilon_+ G^\mu_- - \nabla_\mu \epsilon_- G^\mu_+ - \nabla_\mu \bar{\epsilon}_+ \overline{G}^\mu_- + \nabla_\mu \bar{\epsilon}_- \overline{G}^\mu_+ \right). \tag{68}$$

The theory on the curved surface is obtained by gauging the new rotation group.

- The chiral ring are so operators, that

$$[Q_B, \mathcal{O}] = 0 \tag{69}$$

Their translations preserve cohomology classes

$$\frac{i}{2}(\partial_0 + \partial_1)\mathcal{O} = \{Q_B, [Q_+, \mathcal{O}]\}, \quad \frac{i}{2}(\partial_0 - \partial_1)\mathcal{O} = \{Q_B, [Q_-, \mathcal{O}]\} \tag{70}$$

Morover, one can show in examples that

$$T_{\mu\nu} = \{Q_B, b_{\mu\nu}\} \tag{71}$$

So one can define multiplication in cohomologies by

$$\phi_i \phi_j = \phi_k C^k_{ij} + [Q_B, \Lambda] \tag{72}$$

without reference to a particular points. Using two-point functions on S^2

$$\eta_{ij} = \langle \phi_i \, \phi_j \rangle_0 \tag{73}$$

one can relate structure constants to three point functions on sphere by

$$C_{ijk} = \langle \phi_i \, \phi_j \, \phi_k \rangle_0 = \eta_{il} C^l_{jk} \tag{74}$$

All the same is applicable to twisted chiral ring of A-twisted theory.

- The statement of mirror-symmetry is that
- Standard logic of super-symmetry defines SUSY vacuum states

5.1 A-twist

We consider here non-linear sigma model.

- The shifted fields are

$$\chi^i = \psi^i_-, \quad \chi^{\bar{i}} = \overline{\psi}^{\bar{i}}_+, \quad \rho^{\bar{i}}_z = \overline{\psi}^{\bar{i}}_-, \quad \rho^i_{\bar{z}} = \psi^i_+, \tag{75}$$

With the action

$$S = \int d^2 z g_{i\bar{j}} h^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^{\bar{j}} \sqrt{h} - i g_{i\bar{j}} \rho^{\bar{j}}_z D_{\bar{z}} \chi^i + i g_{i\bar{j}} \rho^i_{\bar{z}} D_z \chi^{\bar{j}} - \frac{1}{2} R_{i\bar{k}j\bar{l}} \rho^i_{\bar{z}} \chi^i \rho^{\bar{k}}_z \chi^{\bar{l}}. \tag{76}$$

- The only remaining SUSY transform got $\bar{\epsilon}_+ = \epsilon_- = 0, \quad \epsilon_+ = \bar{\epsilon}_- = \epsilon$

$$\delta\phi^i = \epsilon_+ \chi^i, \quad \delta\overline{\phi}^{\bar{i}} = \bar{\epsilon}_- \chi^{\bar{i}}, \quad \delta\rho^i_{\bar{z}} = 2i\bar{\epsilon}_- \partial_{\bar{z}} \phi^i + \epsilon_+ \Gamma^i_{jk} \rho^j_{\bar{z}} \chi^k, \quad \delta\rho^{\bar{i}}_z = -2i\epsilon_+ \partial_z \overline{\phi}^{\bar{i}} + \bar{\epsilon}_- \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \rho^{\bar{k}}_z \chi^{\bar{j}}, \quad \delta\chi^i = \delta\chi^{\bar{i}} = 0 \tag{77}$$

- The chiral ring is isomorphic to $H^*_D R(X)$ with elements represented by

$$\omega_{i_1,...,i_p,\bar{j}_1,...,\bar{j}_q}(\phi)\chi^{i_1}...\chi^{i_p}\chi^{\bar{j}_1}...\chi^{\bar{j}_q}, \quad Q_- \to \partial, \quad \overline{Q}_+ \to \bar{\partial} \tag{78}$$

We don't use ρ as we can't create zero-form using it, and not using a metric. The corresponding charges if $\wr \in H^{p_i,q_i}(X)$ are

$$q_V = -p_i + q_i, \quad q_A = p_i + q_i \tag{79}$$

Vector symmetry is not anomalous so $\sum p_i = \sum q_i$. The axial anomaly is

$$\#(\chi \text{ zero mode}) - \#(\rho \text{ zero mode}) = 2\int_\Sigma \phi^* c_1(X) + 2(1-g)\mathrm{dim}X = 2(c_1(X)\cdot\beta) + 2(1-g)\mathrm{dim}X = 2k \tag{80}$$

so

$$\sum p_i = \sum q_i = k \tag{81}$$

- The localization locus is $\partial_{\bar{z}}\phi^i = 0$. The only part of the action, which contributes is

$$\int_\Sigma \phi^*(\omega - iB) \tag{82}$$

The bosonic and fermionic determinant cancel each other, so

$$\langle \mathcal{O}_1 ... \mathcal{O}_s \rangle = e^{-(\omega - iB)\cdot\beta} \int\limits_{\mathcal{M}(X,\beta)} ev_1^*\omega_1 \wedge ... \wedge ev_s^*\omega_s \tag{83}$$

where we take pull-back along the map

$$ev_i : \mathcal{M}_\Sigma(X,\beta) \to X, \quad \phi \mapsto \phi(x_i) \tag{84}$$

Note, that $\omega \cdot \beta \geq 0$ so in the large volume limit, when $\beta = 0$ dominates, all what remain is integral over X

$$\mathcal{M}_\Sigma(X, 0) = X \tag{85}$$

So, from selection rules we deal with genus 0, and the number of operator is so, that

$$\langle \mathcal{O}_1, ..., \mathcal{O}_s \rangle_0 = \int\limits_X \omega_1 \wedge ... \wedge \omega_s = \#(D_1 \cap ... \cap D_s) \tag{86}$$

- The path integral is depending only on twisted-chiral parameters holomorphically, so only on Kahler deformations.

- For the case of $\mathbb{C}P^1$ there is a pair of cohomology classes
  - Operator P corresponding to $1 \in H^0(\mathbb{C}P^2)$, dual to whole $\mathbb{C}P^2$
  - Operator Q corresponding to $H \in H^2(\mathbb{C}P^2)$, dual to point so $\int_{\mathbb{C}P^1} H = 1$
  - The only non-trivial two-point function is $\langle PQ \rangle = 1$
  - The only non-trivial three-point function is $\langle QQQ \rangle = \sum_{n\in\mathbb{Z}} \langle QQQ \rangle_n$.
  - Since $c_1(\mathbb{C}P^1) = 2H$, the axial anomaly for degree n map is $2k$ with $k = c_1(\mathbb{C}P^1)\beta + \dim \mathbb{C}P^1(1-0) = 2n+1$. Since axial charge for Q is two, only degree one map contribute.
  - As the class is Poincare-dual to the class of the point, the correlation function computes number of maps from three points to three points. Thus

$$\langle QQQ \rangle = \langle QQQ \rangle_1 = e^{-t}. \tag{87}$$

  - So the quantum cohomology of $\mathbb{C}P^1$ is

$$PP = P, \quad PQ = QP = Q, \quad QQ = e^{-t}P \tag{88}$$

5.2 B-twist

- We name new fields by

  which are sections of
- The redefined SYSY transformations are given by
- The closed operators are holomorphic functions on X. The Q_B exact terms are those, which can be presented in the form

$$vW = v^i \partial_i W \tag{89}$$

  for some holomorphic vector field v. So the chiral ring is

$$\mathbb{C}[\phi^1, ..., \phi^n]/(\partial_i W) \tag{90}$$

- The action in redefined fields is .. or extracting Q-exact terms ...
- The path integral localizes to

$$\langle O_{f_1} ... O_{f_s} \rangle_g = \sum_{i=1}^{N} f_1(y_i) ... f_s(y_i) (\det \partial_i \partial_j W)^{g-1}(y_i) \tag{91}$$

  where the sum goes over critical points of potential.
- The path integral is depending only on chiral parameters holomorphically.
- The anomaly is

  so the only allowed correlators are
- One can represent
- The mirror of $\mathbb{C}P^1$ is $\mathbb{C}^\times$ with $W = z + e^{-t} z^{-1}$.
  - The chiral ring is $\mathbb{C}[z]/(z^2 - e^{-t})$.
  - The critical points are $z = \pm e^{-t/2}$.
  - The only non-trivial three-point functions in genus zero are

$$\langle 11z \rangle = 1, \quad \langle zzz \rangle = e^{-t} \tag{92}$$

    which coincide with A-model three point functions for $\mathbb{C}P^1$.