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ТВИСТОВАННЫЕ ПРЕДСТАВЛЕНИЯ ТОРОИДАЛЬНЫХ АЛГЕБР И ИХ ПРИМЕНЕНИЯ

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TWISTED REPRESENTATIONS OF TOROIDAL ALGEBRAS AND THEIR APPLICATIONS

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I hereby declare that the work presented in this thesis was carried out by myself at Skolkovo Institute of Science and Technology, Moscow and National Research University ‘Higher School of Economics’, Moscow, and has not been submitted for any other degree.

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Abstract

In this thesis we study representation theory of quantum toroidal algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$. We construct an explicit realization of twisted Fock module $\mathcal{F}_u^{(n',n)}$ identifying it with integrable level 1 representation of quantum affine $\mathfrak{gl}_\mathfrak{n}$ algebra. Chevalley generators of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ expressed via vertex operators of quantum affine $\mathfrak{gl}_\mathfrak{n}$.

We consider particular cases $q_2 = 1$ and $n = 2$. In the first case we generalize our construction and apply it for study of Nekrasov partition functions. Also, in both cases we considered the corresponding deformed twisted W-algebras.

Publications

1. Bershtein, Mikhail; Gonin, Roman. *Twisted representations of algebra of q-difference operators, twisted q-W algebras and conformal blocks.* SIGMA Symmetry Integrability Geom. Methods Appl. 16 (2020), Paper No. 077, 55 pp.

2. Bershtein, Mikhail; Gonin, Roman. *Twisted and non-twisted deformed Virasoro algebras via vertex operators of* $U_q(\widehat{\mathfrak{sl}}_2)$. Lett. Math. Phys. 111 (2021), no. 1, Paper No. 22.

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Contents

Introduction

Preliminary information

In this thesis, we study the representation theory of quantum toroidal $\mathfrak{gl}_1$. This algebra has appeared in different areas of mathematics and mathematical physics independently. The algebra is known as $W_{\infty+1}$-algebra [FHS+10], elliptic Hall algebra [SV13b], double affine Hecke algebra for $\mathfrak{gl}_\infty$ and Ding-Ioharo-Miki algebra [DI97b, Mik07]. We will denote this algebra by $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$. From the geometric representation theory viewpoint, $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ acts on equivariant K-theory of certain moduli spaces of sheaves on $\mathbb{P}^2$ [Neg15a, Tsy17]. Due to different viewpoints, it is very interesting and useful to study $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$.

In the paper [DI97b], algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ has appeared in a list of algebras, interpreted as a Drinfeld double. Due to this interpretation, we automatically obtain all remarkable structures of quantum groups (Hopf algebra structure and R-matrix). This approach yield a presentation of algebra in terms of *Chevalley generators* E_k, F_k for $k \in \mathbb{Z}$ and H_l for $l \neq 0$.

Let us remark that toroidal $\mathfrak{gl}_n$ algebras for $n > 1$ also appear in the list of Drinfeld doubles. Let us denote the algebra by $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_n)$ [Vas98]. The algebras act on K-theory of sheaves on the quotient of $\mathbb{P}^2$ by finite subgroup $\mathbb{Z}/n\mathbb{Z}$ [VV98]. It is surprising that analogous results for $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ were obtained more than 10 years later.

Algebras $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ admit another presentation. It is generated by $P_{a,b}$ for $(a,b) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ and central elements c and c' with certain relations. The Chevalley generators are expressed as follows

$$E_k = P_{1,k} \qquad H_l = P_{0,l} \qquad F_k = P_{-1,k} \tag{0.0.1}$$

This generators and relations originally appeared as a presentation of Hall algebra of an elliptic curve [SV11, SV13b]. A remarkable property of this presentation is an explicit construction of an action of $\widetilde{SL_2}(\mathbb{Z})$ on $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$. Here $\widetilde{SL_2}(\mathbb{Z})$ is a central extension of $SL_2(\mathbb{Z})$ by $\mathbb{Z}$. Group $SL_2(\mathbb{Z})$ acts on the quotient of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ modulo $c = c' = 1$, the action is given by the formula $\sigma P_{a,b} = P_{\sigma(a,b)}$. Another remarkable result is PBW-type theorem for generators $P_{a,b}$.

DAHA and Macdonald polynomials Double affine Hecke algebra $\mathcal{H}_N$ (abbreviated as DAHA) acts on the space of Laurent polynomials $\mathbb{C}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$. Let Λ_N be the ring of symmetric Laurent polynomials in N variables. There is a spherical subalgebra in $\mathcal{H}_N$, denote it by $S\mathcal{H}_N$; we also will refer to the subalgebra as *spherical DAHA*. Spherical DAHA $S\mathcal{H}_N$ acts on Λ_N. There is a commutative subalgebra in $S\mathcal{H}_N$ acting by diagonalizable operators. The operators are called *Macdonald operators*, and eigenvectors are *Macdonald polynomials* [Che92, Kir97, Mac03].

Let Λ be the ring of symmetric function in infinitely many variables. Then an analogue of $S\mathcal{H}_N$ is $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ [FFJ+11a, SV11, SV13b]. Operators $P_{a,0}$ are Macdonald operators. For $b > 0$, operators $P_{0,b}$ are the operators of multiplication by power sum symmtric polynomial p_b (up to a normalization). Operators $P_{0,-b}$ act as $\partial/\partial p_b$ (up to a normalization). Algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ is generated by $P_{0,b}$ and $P_{a,0}$, hence action of the operators determines action of whole $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ on Λ. The representation obtained is called *Fock module* $\mathcal{F}_u$. The parameter u appears due to an automorphism $P_{a,b} \mapsto u^a P_{a,b}$.

In Macdonald theory, the polynomial depend on parameters q, t. The parameters are connected with our parameters as follows $q_1 = q$, $q_3 = t^{-1}$.

Bosonization There is another construction of Fock module [FHH+09]. Consider a Heisenberg algebra generated by a_k and the corresponding Fock space F^a. There is an action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ on F^a determined by explicit formulas for Chevalley generators in terms of a_k. More precisely, H_n acts as the generators a_k (up to normalization), and for $E(z) = \sum_k E_k z^{-k}$ and $F(z) = \sum_k F_k z^{-k}$ there is an explicit formula as a normally ordered exponent of a_k.

Such constructions are called *bosonization.* Bosonization is an efficient tool for study Kac-Moody algebras, and, more generally, in Conformal Field Theory [DFMS97].

It is remarkable that comultiplication gives explicit bosonization of $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_n}$. The action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ is expressed via n copies of Heisenberg algebra. Formulas for $E(z)$ and $F(z)$ is given by a sum n normally ordered exponents.

Deformed W-algebras W-algebras appeared in Conformal Field Theory as a generalization of Virasoro algebra. Then deformed W-algebras $\mathcal{W}_{q_1,q_2}(\mathfrak{sl}_n)$ appeared in [FF96]. The original definition of W was not via generators and relations, but via a bosonization, i.e. as an algebra, generated by certain operators, expressed via Heisenberg algebra. A particular case of deformed W-algebras is deformed Virasoro algebra $\mathcal{W}_{q_1,q_2}(\mathfrak{sl}_2)$, which was originally defined via generators and relations [SKAO96].

Using the explicit bosonization formulas one can see that Chevalley generators of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ act on $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_n}$ as generators of $\mathcal{W}_{q_1,q_2}(\mathfrak{gl}_n) = \mathcal{W}_{q_1,q_2}(\mathfrak{sl}_n) \oplus \mathsf{Heis}$, here Heis is a Heisenberg algebra. In other words, $\mathcal{W}_{q_1,q_2}(\mathfrak{gl}_n) = U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)/I_n$, here I_n is a two-sided ideal, annihilated by $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_n}$ [FHS+10].

Many important ingredients of Conformal Field Theory, including conformal blocks, can be defined in terms of W-algebras. The algebras $\mathcal{W}_{q_1,q_2}(\mathfrak{sl}_n)$ determine a q-deformation of conformal blocks [AFO18].

K-theory of moduli spaces Let $\mathcal{M}_{n,k}$ be the moduli space of torsion free sheaves on $\mathbb{P}^2$ of rank n and with second Chern class k and fixed trivialization at infinity. Consider a torus $T = \mathbb{C}^*_{q_1} \times \mathbb{C}^*_{q_2} \times \mathbb{C}^*_{u_1} \times \cdots \times \mathbb{C}^*_{u_n}$. There is an action $T \curvearrowright \mathcal{M}_{n,k}$, induced from the tautological action $\mathbb{C}^*_{q_1} \times \mathbb{C}^*_{q_2} \curvearrowright \mathbb{P}^2$ and action of $\mathbb{C}^*_{u_1} \times \cdots \times \mathbb{C}^*_{u_n}$ by changing trivialization at infinity. Denote by $K_T(\mathcal{M}_{n,k})_{\mathrm{loc}}$ localized equivariant K-theory $\mathcal{M}_{n,k}$ with respect to the action of T. Finally, let

$$\mathbf{K}_n = \bigoplus_{k=0}^{\infty} K_T(\mathcal{M}_{n,k})_{\mathrm{loc}} \tag{0.0.2}$$

Action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ on $\mathbf{K}_n$ was constructed via correspondences. The obtained representation is isomorphic to $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_n}$ [Neg15a, Tsy17], this is a generalization of results of Nakajima [Nak97].

Tor T acts on $\mathcal{M}_{n,k}$ with finitely many fixed points. Due to localization theorem $K_T(\mathcal{M}_{n,k})_{\mathrm{loc}}$ has a basis, enumerated by the fixed points. The vectors are eigenvectors of $P_{a,0}$. In particular, for $n = 1$ the obtained basis is Macdonald basis with respect to the identification $\mathbf{K}_1 = \Lambda$

All this is particularly interesting in light of Alday, Gaiotto, and Tachikawa conjecture about a correspon-dence between supersymmetric gauge theories and conformal field theories [AGT10]. The correspondence can be formulated as a mathematical statement about action of $\mathcal{W}_{q_1,q_2}(\mathfrak{gl}_n)$ on $\mathbf{K}_n$ [Neg18]. Also, all this admits a yangian version with K theory replaced by cohomology [MO19, SV13a].

Gorsky-Neguț conjecture K-theoretic stable envelopes form an important basis in a symplectic resolution of X. The basis depend on an additional parameter called slope $s \in \mathrm{Pic}(X) \otimes \mathbb{R}\backslash\{\text{walls}\}$;

here 'walls' is a hyperplane arrangement. Actually the basis is determined by connection component of $\mathrm{Pic}(X) \otimes \mathbb{R} \backslash \{\text{walls}\}$. It is interesting to study change of basis, corresponding to wall-crossing.

An important example of symplectic resolution X is $\mathcal{M}_{n,k}$. Moreover, we can consider case $n = 1$, then $\mathcal{M}_{1,k}$ is Hilbert scheme of k point $\mathbb{C}^2$, denoted by $\mathrm{Hilb}_k(\mathbb{C}^2)$. In this case $\mathrm{Pic}(X) = \mathbb{Z}$; the walls are a/b for $b \leq k$.

E. Gorsky and A. Neguț studied the change of basis on the walls m/n [GN17]. They have conjectured an answer in terms of $U_v(\widehat{\mathfrak{gl}}_n)$. Recall that $U_v(\widehat{\mathfrak{gl}}_n)$ has n integrable level 1, denote them by $F_0, \dots, F_{n-1}$. The conjecture says that there is an isomorphism $\mathbf{K}_1 \xrightarrow{\sim} F_i$ such that stable envelopes with the slopes $s = m/n \pm \epsilon$ are mapped to standard and costandard bases of F_i correspondingly. Here $\pm\epsilon$ are infinitesimal shifts away from the wall. The conjecture was recently proved by Y. Kononov and A. Smirnov [KS20a].

Twisted Fock modules Let M be a representation of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$, $\tau \in \widetilde{SL}(2,\mathbb{Z})$. Twisted representation M^τ coincides with M as a vector space, but the action is twisted by the automorphism τ. Let $p(\tau) \in SL(2,\mathbb{Z})$ be the projection of τ. Denote

$$p(\tau) = \begin{pmatrix} m' & m \\ n' & n \end{pmatrix}. \tag{0.0.3}$$

In this thesis, we will study twisted Fock modules $\mathcal{F}_u^\tau$. The representation $\mathcal{F}_u^\tau$ is essentially determined by (n', n). More precisely, another choice of τ for a fixed (n', n) corresponds to a shift of u. In particular, for $m' = m = n = 1$ and $n' = 0$, twisted Fock module is isomorphic to Fock module. The intertwiner operator ∇ is called Bergon, Garsia, Haiman operator [BGHT99] and plays an important role in combinatorics of Macdonald polynomials.

From the point of view of explicit realization, change of parameters $(n', n) \mapsto (n' + nk, n)$ corresponds to $E(z) \mapsto z^k E(z)$ and $F(z) \mapsto z^{-k} F(z)$. Informally speaking, it is interesting to study the dependence on n and residue of n' modulo n. Note that, residue of n' is determined by residue of m since $n'm = 1 \bmod n$.

Note that $\tau P_{k,0}$ equals $P_{km,kn}$ up a monomial in c and c'. In paper [Neg16a] the author has calculated action of $P_{km,kn}$ in basis of stable envelopes with slope $m/n - \epsilon$. The action of Heisenberg algebra corresponding to scaling matrices in $U_v(\widehat{\mathfrak{gl}}_n)$ on standard basis of F_0 has exactly the same form. This observation suggests that there is a connection between stable envelopes for slope $m/n \pm \epsilon$ and twisted Fock module $\mathcal{F}_u^\tau$. This connection was one of the motivations for Gorsky-Neguț conjecture. Also, let us remark that twisted Fock modules were used for calculations in topological strings [AFS12] and knot theory [GN15].

Thesis results

In this thesis, we study explicit realizations of twisted Fock modules of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ and twisted W-algebras.

- In case $q_2 = 1$, we have constructed three realizations of twisted Fock module $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$: fermionic (Theorem 0.0.1), bosonic and strange bosonic (Theorem 0.0.3). It was proved that $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ acts via a quotient, isomorphic to twisted deformed W-algebra (Theorems 0.0.4 and 0.0.5). These results were generalized for representation obtained by *restriction to a sublattice*. As an application, we have proved an identity for q-deformed conformal blocks.

- We have constructed explicitly action of twisted and non-twisted Virasoro algebras on an integrable level 1 representation of quantum affine $\mathfrak{sl}_2$ (Theorems 0.0.6 and 0.0.7 correspondingly). The answer is expressed via vertex operators of quantum affine $\mathfrak{sl}_2$.

- We have constructed explicitly twisted Cherednik representation of double affine Hecke algebra $\mathcal{H}_N$ (Theorems 0.0.8 and 0.0.9). Twisted Fock module of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ is constructed explicitly via semi-infinite construction (Theorem 0.0.10 and 0.0.12). Action of Chevalley generators is expressed via vertex operators $U_v(\widehat{\mathfrak{gl}}_n)$. As a corollary, we have constructed an identification (as vector spaces) of representations F_i and $\mathcal{F}_u^{\tau}$.

Below we will formulate the results of the thesis in more details. Also we will give links to the main part of the thesis. We hope, this will help a reader to navigate.

Schur specialization case

Below we will review some results of Chapter 1.

In this section we consider specialization $q_2 = 1$ of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$. It is convient to introduce a parameter $q = q_3 = q_1^{-1}$. The specialization is called *Schur specialization* since Macdonald polynomials become Schur polynomials in this case.

Definition 0.0.1. *Lie algebra $\mathfrak{Diff}_q$ has a basis $E_{k,l}$ (for $(k,l) \in \mathbb{Z}^2 \backslash \{(0,0)\}$), $\mathbf{c}$ and $\mathbf{c}'$. Elements $\mathbf{c}$ and $\mathbf{c}'$ are central. All other commutators are given by the following formula*

$$[E_{k,l}, E_{r,s}] = (q^{\frac{ks-lr}{2}} - q^{\frac{lr-ks}{2}})E_{k+r,l+s} + \left(l\mathbf{c} + k\mathbf{c}'\right)\delta_{k+r,0}\delta_{l+s,0} \tag{0.0.4}$$

We will define the algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ below, but we want to formulate the relation between $\mathfrak{Diff}_q$ and $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$.

Proposition 0.0.1. *Universal enveloping of $\mathfrak{Diff}_q$ is a specialization $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ for $q = q_3 = q_1^{-1}$. The isomorphism is given by the formula $E_{a,b} = q^{d/2}P_{a,b}$, for $d = \gcd(a,b)$. Moreover*

$$\mathbf{c} = 2\lim_{q_2\to 1}\frac{c-1}{q_2-1} \qquad \mathbf{c}' = 2\lim_{q_2\to 1}\frac{c'-1}{q_2-1} \tag{0.0.5}$$

Let us determine the following currents

$$\mathbf{E}(z) = \sum_{j\in\mathbb{Z}} E_{1,j}z^{-j} \qquad \mathbf{F}(z) = \sum_{j\in\mathbb{Z}} E_{-1,j}z^{-j} \tag{0.0.6}$$

Consider Heisenberg algebra $[\mathbf{a}_k, \mathbf{a}_l] = k\delta_{k+l,0}$. Denote its Fock space by $F^{\mathbf{a}}$.

Proposition 0.0.2 ([GKL92])**.** *The following formulas determine an action of $\mathfrak{Diff}_q$ on $F^{\mathbf{a}}$*

$$\mathbf{c} \mapsto 1, \qquad \mathbf{c}' \mapsto 0 \qquad E_{0,j} \mapsto \mathbf{a}_j, \tag{0.0.7}$$

$$\mathbf{F}(z) = \frac{u^{-1}}{1-q^{-1}}\exp\left(\sum_{k>0}\frac{q^{k/2}-q^{-k/2}}{k}\mathbf{a}_{-k}z^k\right)\exp\left(\sum_{k>0}\frac{q^{k/2}-q^{-k/2}}{k}\mathbf{a}_k z^{-k}\right) \tag{0.0.8}$$

Denote the obtained representation by $\mathscr{F}_u$.

Action of $SL_2(\mathbb{Z})$ and twisted rerepesentations

Let σ be an element of $SL_2(\mathbb{Z})$

$$\sigma = \begin{pmatrix} m' & m \\ n' & n \end{pmatrix}. \tag{0.0.9}$$

There is an action of $SL_2(\mathbb{Z})$ on $\mathfrak{Diff}_q$, determined by the following formulas

$$\sigma(E_{k,l}) = E_{m'k+ml,\, n'k+nl}, \quad \sigma(c') = m'c' + n'c, \quad \sigma(c) = mc' + nc. \tag{0.0.10}$$

For any $\mathfrak{Diff}_q$-module M, let $\rho_M \colon \mathfrak{Diff}_q \to \mathfrak{gl}(M)$ be the corresponding homomorphism.

Definition 0.0.2. *For any $\mathfrak{Diff}_q$-module M and $\sigma \in SL(2,\mathbb{Z})$ let us define M^σ as follows. M and M^σ are the same vector space with different actions of $\mathfrak{Diff}_q$, namely $\rho_{M^\sigma} = \rho_M \circ \sigma$.*

We refer to M^σ as *twisted representation.*

Explicit constructions

Fermionic construction Consider an algebra, generated by $\psi_{(a)}[i]$ and $\psi^*_{(b)}[j]$ for $i, j \in \mathbb{Z}$; $a, b = 0, \ldots, n-1$, satisfying the following relations

$$\left\{\psi_{(a)}[i], \psi_{(b)}[j]\right\} = 0; \quad \left\{\psi^*_{(a)}[i], \psi^*_{(b)}[j]\right\} = 0; \tag{0.0.11}$$

$$\left\{\psi_{(a)}[i], \psi^*_{(b)}[j]\right\} = \delta_{a,b}\ \delta_{i+j,0}. \tag{0.0.12}$$

Consider currents

$$\psi_{(a)}(z) = \sum_i \psi_{(a)}[i] z^{-i-1}; \quad \psi^*_{(b)}(z) = \sum_i \psi^*_{(b)}[i] z^{-i}. \tag{0.0.13}$$

Consider a module $F^{n\psi}$ with cyclic vector $|l_0, \ldots, l_{n-1}\rangle$ and relations

$$\psi_{(a)}[i]\ |l_0, \ldots, l_{n-1}\rangle = 0 \quad \text{for } i \geqslant l_a, \tag{0.0.14}$$

$$\psi^*_{(a)}[j]\ |l_0, \ldots, l_{n-1}\rangle = 0 \quad \text{for } j > -l_a. \tag{0.0.15}$$

Let us define a grading $\deg_{\mathbf{a}_0}$ as follows

$$\deg_{\mathbf{a}_0} |l_0, \ldots, l_{n-1}\rangle = -\, l_0 - \cdots - l_{n-1} \qquad \deg_{\mathbf{a}_0} \psi_{(a)}[i] = 1 \qquad \deg_{\mathbf{a}_0} \psi^*_{(b)}[i] = -\, 1 \tag{0.0.16}$$

Let $F^{n\psi}_m$ be a component of degree m.

Theorem 0.0.1 (Theorem 1.4.1)**.** *For $n \in \mathbb{Z}_{>0}$ and $n_{tw} \in \mathbb{Z}$ the following formulas determine an action of $\mathfrak{Diff}_q$ on $F^{n\psi}_0$*

$$\mathbf{c}' = n_{tw}, \quad \mathbf{c} = n, \tag{0.0.17}$$

$$E_{0,k} = \sum_a \sum_{i+j=k} \psi_a[i]\ \psi^*_a[j], \tag{0.0.18}$$

$$\mathbf{E}(z) = \sum_{b-a \equiv -n_{tw} \bmod n} u^{\frac{1}{n}} q^{-1/2} z \psi_{(a)}(q^{-1/2}z) \psi^*_{(b)}(q^{1/2}z) z^{\frac{n_{tw}-a+b}{n}} q^{(a+b)/2n}, \tag{0.0.19}$$

$$\mathbf{F}(z) = \sum_{b-a \equiv n_{tw} \bmod n} u^{-\frac{1}{n}} q^{1/2} z \psi_{(a)}(q^{1/2}z) \psi^*_{(b)}(q^{-1/2}z) z^{\frac{-n_{tw}-a+b}{n}} q^{-(a+b)/2n}. \tag{0.0.20}$$

Recall (0.0.9)*. For $n_{tw} = n'$ the representation obtained is isomorphic to $\mathscr{F}^\sigma_u$.*

This gives us an explicit construction of $\mathscr{F}^\sigma_u$. Moreover, we have obtained explicit constructions of more general representations $\mathscr{F}^{(n_{tw},n)}_u$. One can define and construct explicitly $\mathscr{F}^{(n_{tw},n)}_u$ for $n < 0$, using σ for $n' = m = 0$ and $n = m' = -1$.

Let $d = \gcd(n_{tw}, n)$. Denote $\mathbf{Q}_{(d)} = \{(l_0, \ldots, l_{d-1}) \in \mathbb{Z}^n | \sum_{i=0}^{d-1} l_i = 0\}$.

Theorem 0.0.2 (Theorem 1.8.1)**.** *The following $\mathfrak{Diff}_q$-modules are isomorphic.*

$$\mathscr{F}^{(n_{tw},n)}_{u^d} \cong \bigoplus_{l \in \mathbf{Q}_{(d)}} \mathscr{F}^{(n_{tw}/d,\, n/d)}_{uq^{l_0}} \otimes \cdots \otimes \mathscr{F}^{(n_{tw}/d,\, n/d)}_{uq^{\frac{\alpha}{d}+l_\alpha}} \otimes \cdots \otimes \mathscr{F}^{(n_{tw}/d,\, n/d)}_{uq^{\frac{d-1}{d}+l_{d-1}}} \tag{0.0.21}$$

Strange bosonic construction Let ζ be nth primitive root of unity, e.g. $\zeta = e^{\frac{2\pi i}{n}}$.

Theorem 0.0.3 (Theorem 1.4.3)**.** *For $n \in \mathbb{Z}_{>0}$ and $n_{tw} \in \mathbb{Z}$ there is an action of $\mathfrak{Diff}_q$ on $F^{\mathbf{a}}$ determined by the following formulas*

$$E_{0,k} = a_{nk}, \quad \mathbf{c} = n, \quad \mathbf{c}' = n_{tw}, \tag{0.0.22}$$

$$E(z) = z^{n_{tw}/n} \frac{u^{\frac{1}{n}}}{n(1-q^{1/n})} \sum_{l=0}^{n-1} \zeta^{l n_{tw}} :\exp\left(\sum_k \frac{q^{-k/2n} - q^{k/2n}}{k} \mathbf{a}_k \zeta^{-kl} z^{-k/n}\right):, \tag{0.0.23}$$

$$F(z) = z^{-n_{tw}/n} \frac{u^{-\frac{1}{n}}}{n(1-q^{-1/n})} \sum_{l=0}^{n-1} \zeta^{-l n_{tw}} :\exp\left(\sum_k \frac{q^{k/2n} - q^{-k/2n}}{k} \mathbf{a}_k \zeta^{-kl} z^{-k/n}\right): . \tag{0.0.24}$$

The representation obtained is isomorphic to $\mathscr{F}_u^{(n_{tw},n)}$.

Twisted q-W-algebra

Below we will continue review of Chapter 1.

In this section we will continue to use specialization $q = q_3 = q_1^{-1}$. Though we expect that the results of this section can be directly generalized any values of q_1, q_2, and q_3.

Definitions Twisted q-W-algebra depends on n and residue of n_{tw} modulo n. For $n_{tw} = 0$, we obtain non-twisted q-W-algebra [FF96]. Algebra $\mathcal{W}_{q,p}(\mathfrak{sl}_2, 1)$ was defined in [Shi04, (37)–(38)].

Let us introduce the following notation

$$\sum_{l=0}^{\infty} f_{k,n}[l] x^l = f_{k,n}(x) = \frac{(1-qx)^{\frac{n-k}{n}} (1-q^{-1}x)^{\frac{n-k}{n}}}{(1-x)^{\frac{2(n-k)}{n}}} \tag{0.0.25}$$

Definition 0.0.3. *Algebra $\mathcal{W}_q(\mathfrak{sl}_n, n_{tw})$ is generated by $T_k^{tw}[r]$ for $r \in n_{tw}k/n + \mathbb{Z}$ and $k = 1, \ldots, n-1$. It is convenient to denote $T_0^{tw}[r] = T_n^{tw}[r] = \delta_{r,0}$. The defining relations are the following*

$$\sum_{l=0}^{\infty} f_{k,n}[l] \Big(T_1^{tw}[r-l] T_k^{tw}[s+l] - T_k^{tw}[s-l] T_1^{tw}[r+l] \Big) = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 (kr - s) T_{k+1}^{tw}[r+s], \tag{0.0.26}$$

$$\sum_{l=0}^{\infty} f_{n-k,n}[l] \Big(T_{n-1}^{tw}[r-l] T_k^{tw}[s+l] - T_k^{tw}[s-l] T_{n-1}^{tw}[r+l] \Big) = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 \big((n-k)r - s\big) T_{k-1}^{tw}[r+s]. \tag{0.0.27}$$

Let us rewrite the relations (0.0.26)–(0.0.27) in current form. Denote

$$T_k^{tw}(z) := \sum T_k^{tw}[r] z^{-r}, \quad T_k(z) := z^{\frac{k n_{tw}}{n}} T_k^{tw}(z), \tag{0.0.28}$$

$$T_k^{\circ}(z) := z^{-\frac{(n-k)n_{tw}}{n}} T_k^{tw}(z) = z^{-n_{tw}} T_k(z). \tag{0.0.29}$$

Note that

$$T_0(z) = T_n^{\circ}(z) = 1 \quad T_n(z) = T_0^{\circ}(z) = z^{n_{tw}}. \tag{0.0.30}$$

Proposition 0.0.3. *The relation* (0.0.26) *is equivalent to*

$$\begin{aligned} &f_{k,n}(w/z) T_1(z) T_k(w) - f_{k,n}(z/w) T_k(w) T_1(z) = \\ &\qquad = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 \left((k+1) \frac{w}{z} \delta'(w/z) T_{k+1}(w) + w \delta(w/z) \partial_w T_{k+1}(w) \right). \end{aligned} \tag{0.0.31}$$

The relation (0.0.27) *is equivalent to*

$$f_{n-k,n}(w/z)T^{\circ}_{n-1}(z)T^{\circ}_{k}(w) - f_{n-k,n}(z/w)T^{\circ}_{k}(w)T^{\circ}_{n-1}(z) = \\ -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 \left((n-k+1)\frac{w}{z}\delta'(w/z)T^{\circ}_{k-1}(w) + w\delta(w/z)\partial_w T^{\circ}_{k-1}(w)\right) \quad (0.0.32)$$

Connection of W-algebras and $\mathfrak{Diff}_q$ A connection between $\mathcal{W}_q(\mathfrak{sl}_n)$ and $\mathfrak{Diff}_q$ is known (see [FHS+10, Prop. 2.14] and [Neg18, Prop. 2.25]). In this section, we will generalize the results for any n_{tw}.

Let $\mathfrak{Heis}$ be Heisenberg algebra, generated by $\tilde{H}_j$ with the relation $[\tilde{H}_i, \tilde{H}_j] = ni\delta_{i+j,0}$. We will prove that there is a surjective homomorphism $\mathfrak{Diff}_q \twoheadrightarrow \mathcal{W}_q(\mathfrak{sl}_n, n_{tw}) \otimes U(\mathfrak{Heis})$. The generators $E_{0,j}$ are mapped to $\tilde{H}_j$.

Let us introduce the following notation

$$\varphi_-(z) = \sum_{j>0} \frac{q^{-j/2} - q^{j/2}}{j} E_{0,-j} z^j, \qquad \varphi_+(z) = -\sum_{j>0} \frac{q^{j/2} - q^{-j/2}}{j} E_{0,j} z^{-j}, \quad (0.0.33)$$

$$\tilde{\varphi}_-(z) = \sum_{j>0} \frac{q^{-j/2} - q^{j/2}}{j} \tilde{H}_{-j} z^j, \qquad \tilde{\varphi}_+(z) = -\sum_{j>0} \frac{q^{j/2} - q^{-j/2}}{j} \tilde{H}_{j} z^{-j}. \quad (0.0.34)$$

Also denote

$$\varphi(z) = \varphi_-(z) + \varphi_+(z), \qquad \tilde{\varphi}(z) = \tilde{\varphi}_-(z) + \tilde{\varphi}_+(z). \quad (0.0.35)$$

Let

$$\tilde{T}_k(z) = \frac{1}{k!} \exp\left(-\frac{k}{n}\varphi_-(z)\right) \mathbf{E}^k(z) \exp\left(-\frac{k}{n}\varphi_+(z)\right). \quad (0.0.36)$$

Note that $\tilde{T}_k(z)$ commutes with H_j.

Let $J_{\mu,n,n_{tw}}$ be a two-sided ideal in $\mathfrak{Diff}_q$, generated by $\mathbf{c} - n$, $\mathbf{c}' - n_{tw}$ and $\tilde{T}_n(z) - \mu^n z^{n_{tw}}$ (here $\mu \in \mathbb{C}\backslash\{0\}$). Parameter μ is not essential since the automorphism $E_{a,b} \mapsto \mu^{-a} E_{a,b}$ maps $J_{\mu,n,n_{tw}}$ to $J_{1,n,n_{tw}}$. We will abbreviate $J_{n,n_{tw}} = J_{\mu,n,n_{tw}}$.

Lemma 0.0.4. *$\tilde{T}_k(z) \in J_{n,n'}$ for $k > n$.*

Theorem 0.0.4 (Theorem 1.7.1)**.** *There is an isomorphism $\mathcal{S}\colon \mathcal{W}_q(\mathfrak{sl}_n, n_{tw}) \otimes U(\mathfrak{Heis}) \xrightarrow{\sim} U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$ determined by the following formulas*

$$T_k(z) \mapsto \mu^{-k}\tilde{T}_k(z); \quad \tilde{H}_j \mapsto E_{0,j} \quad (0.0.37)$$

The inverse map $\mathcal{P}$ is determined as follows

$$E_{0,j} \mapsto \tilde{H}_j; \quad \mathbf{c} \mapsto n; \quad \mathbf{c}' \mapsto n_{tw}; \quad (0.0.38)$$

$$\mathbf{E}(z) \mapsto \ \mu \exp\left(\frac{1}{n}\tilde{\varphi}_-(z)\right) T_1(z) \exp\left(\frac{1}{n}\tilde{\varphi}_+(z)\right) \quad (0.0.39)$$

$$\mathbf{F}(z) \mapsto -\frac{\mu^{-1} z^{-n_{tw}}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2} \exp\left(-\frac{1}{n}\tilde{\varphi}_-(z)\right) T_{n-1}(z) \exp\left(-\frac{1}{n}\tilde{\varphi}_+(z)\right) \quad (0.0.40)$$

Theorem 0.0.5 (Theorem 1.7.2)**.** *The ideal $J_{\mu,nd,n_{tw}d}$ is annihilated by $\mathcal{F}^{(n_{tw},n)}_{u_1} \otimes \cdots \otimes \mathcal{F}^{(n_{tw},n)}_{u_d}$ for*

$$\mu = (-1)^{\frac{1}{n}} \frac{q^{-\frac{1}{2n}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} (u_1 \cdots u_d)^{\frac{1}{nd}}, \quad (0.0.41)$$

As a corollary, we obtain an action of $\mathcal{W}(\mathfrak{sl}_{nd}, n_{tw}d)$ on $\mathcal{F}^{(n_{tw},n)}_{u_1} \otimes \cdots \otimes \mathcal{F}^{(n_{tw},n)}_{u_d}$. Using Theorems 0.0.4 and 0.0.5 we obtain an explicit realization of $\mathcal{W}_q(\mathfrak{sl}_n, n_{tw})$ from fermionic realization (Theorem 0.0.1) and strange bosonic realization (Theorem 0.0.3).

Twisted and non-twisted Virasoro algebra

Algebra $\mathcal{W}_q(\mathfrak{sl}_n, n_{tw})$ is a specialization of $\mathcal{W}_{q_1,q_2}(\mathfrak{sl}_n, n_{tw})$ for $q = q_3 = q_1^{-1}$. We expect that the results of Section *Twisted q-W-algebra* can be generalized for $q_2 \neq 1$. Therefore it is interesting to construct explicit realization $\mathsf{Vir}_{q_1,q_2} = \mathcal{W}_{q_1,q_2}(\mathfrak{sl}_2, 0)$ and $\mathsf{Vir}^{\mathrm{tw}}_{q_1,q_2} = \mathcal{W}_{q_1,q_2}(\mathfrak{sl}_2, 1)$.

Bosonization of $U_q(\widehat{\mathfrak{sl}}_2)$ and its vertex operators

We recall bosonization of $U_q(\widehat{\mathfrak{sl}}_2)$ and the corresponding vertex operators. All this can be found in [JM95, Chapters 5,6]. Our notation almost coincide with notation from [JM95]; though we use different noramlization of vertex operators.

Bosonization. Algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is generated by $x^{\pm}_k$, a_l for $k \in \mathbb{Z}$, $l \in \mathbb{Z}_{\neq 0}$, $K^{\pm 1}$ and central elements $\gamma^{\pm 1/2}$. These generators are called *Drinfeld generators.* One can find defining relaitons in [JM95, (5.3)–(5.7)], though let us recall

$$[a_k, a_l] = \delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{v - v^{-1}}, \tag{0.0.42}$$

here $[n] = (v^n - v^{-n})/(v - v^{-1})$.

Let Λ_0, Λ_1 be fundamental weights of the algebra $\widehat{\mathfrak{sl}}_2$ and let α be the positive root of $\mathfrak{sl}_2 \subset \widehat{\mathfrak{sl}}_2$. We will construct two representations F_0 and F_1 of $U_q(\widehat{\mathfrak{sl}}_2)$. As vector spaces

$$F_i = \mathbb{C}[a_{-1}, a_{-2}, \dots] \otimes \left(\oplus_n \mathbb{C}e^{\Lambda_i + n\alpha}\right). \tag{0.0.43}$$

As the representations of Heisenberg algebra a_k, these modules are countable sum of Fock spaces

$$V_j = \mathbb{C}[a_{-1}, a_{-2}, \dots] \otimes \mathbb{C}e^{\Lambda_i + \lfloor \frac{j}{2} \rfloor \alpha} \quad \text{for } i \equiv j \bmod 2. \tag{0.0.44}$$

Let us define operators $e^{\pm\alpha}$ and ∂ as follows

$$e^{\pm\alpha}\left(f \otimes e^{\beta}\right) = f \otimes e^{\beta \pm \alpha}, \qquad \partial\left(f \otimes e^{\beta}\right) = (\alpha, \beta) f \otimes e^{\beta}. \tag{0.0.45}$$

Action of $U_q(\widehat{\mathfrak{sl}}_2)$ is given by the following formulas

$$K = v^{\partial}, \quad \gamma = v, \tag{0.0.46}$$

$$X^{\pm}(z) = \exp\left(\pm\sum_{n=1}^{\infty}\frac{a_{-n}}{[n]}v^{\mp n/2}z^n\right)\exp\left(\mp\sum_{n=1}^{\infty}\frac{a_n}{[n]}v^{\mp n/2}z^{-n}\right)e^{\pm\alpha}z^{\pm\partial}, \tag{0.0.47}$$

here $X^{\pm}(z) = \sum x^{\pm}_k z^{-k-1}$. The representation obtained is irreducible highest weight representation with the highest vector $|\Lambda_i\rangle = 1 \otimes e^{\Lambda_i} \in F_i$.

Vertex operators. Let V_z be the evaluation representation $U_q(\widehat{\mathfrak{sl}}_2)$, corresponding to standard two-dimensional representaiton of $\mathfrak{sl}_2$. Vertex operators of $U_q(\widehat{\mathfrak{sl}}_2)$ are intertwining operators

$$\Phi^{(i,i-1)}(z) \colon F_i \to F_{i-1} \otimes V_z, \qquad \Psi^{(i,i-1)}(z) \colon F_i \to V_z \otimes F_{i-1}, \tag{0.0.48}$$

$$\Phi^{*(i,i+1)}(z) \colon F_i \otimes V_z \to F_{i+1}, \qquad \Psi^{*(i,i+1)}(z) \colon V_z \otimes F_i \to F_{i+1}. \tag{0.0.49}$$

Below we will abbreviate $\Phi(z) = \Phi^{(1-i,i)}(z)$, if a statement holds for both $i = 0, 1$.

Let e_+ and e_- be standard basis in the two-dimensional representation. Let us define operators $\Phi_\pm(z)$ and $\Phi^*_\pm(z)$ as follows

$$\Phi(z)\mathbf{w} = \Phi_+(z)\mathbf{w}\otimes e_+ + \Phi_-(z)\mathbf{w}\otimes e_- \tag{0.0.50}$$
$$\Phi^*(z)(\mathbf{w}\otimes e_+) = \Phi_+(z)\mathbf{w} \qquad \Phi^*(z)(\mathbf{w}\otimes e_-) = \Phi_-(z)\mathbf{w} \tag{0.0.51}$$

Analogously, we can define $\Psi_\pm(z)$ and $\Psi^*_\pm(z)$. The operators exist and unique up to normalization. Moreover, there are explicit formulas for $\Phi_-(z)$ and $\Psi_+(z)$ [JM95]. Below we recall the formulas (also the formulas fix the normalization)

$$\Phi_-(z) = \exp\left(\sum_{n=1}^\infty \frac{a_{-n}}{[2n]}v^{7n/2}z^n\right)\exp\left(-\sum_{n=1}^\infty \frac{a_n}{[2n]}v^{-5n/2}z^{-n}\right)e^{\alpha/2}(-v^3z)^{\partial/2}, \tag{0.0.52}$$
$$\Psi_+(z) = \exp\left(-\sum_{n=1}^\infty \frac{a_{-n}}{[2n]}v^{n/2}z^n\right)\exp\left(\sum_{n=1}^\infty \frac{a_n}{[2n]}v^{-3n/2}z^{-n}\right)e^{-\alpha/2}(-vz)^{-\partial/2}. \tag{0.0.53}$$

The operators $\Phi_+(z)$, $\Psi_-(z)$ are given by the following formulas (this follows directly from the definition as an intertwiner)

$$\Phi_+(z) = [\Phi_-(z), x_0^-]_v, \qquad\qquad \Psi_-(z) = [\Psi_+(z), x_0^+]_v. \tag{0.0.54}$$

here we use the notation $[A,B]_p = AB - pBA$. Dual operators are given by the following formulas $\Phi^*_\varepsilon(z) = \Phi_{-\varepsilon}(v^{-2}z)$ and $\Psi^*_\varepsilon(z) = \Psi_{-\varepsilon}(v^2z)$

Deformed Virasoro algebra

Below we will consider algebra Vir_{q_1,q_2} depending on two parameters q_1, q_2. Also we set $v = q_2^{-1/2}$. To define the algebra Vir_{q_1,q_2}, we will need the following notation

$$\sum_{l=0}^\infty f_l x^l = f(x) = \exp\left(\sum_{n=1}^\infty \frac{1}{n}\frac{(1-q_1^n)(1-q_3^n)}{1+q_2^{-n}}x^n\right). \tag{0.0.55}$$

Definition 0.0.4. *Deformed Virasoro algebra Vir_{q_1,q_2} is generated by T_n for $n\in\mathbb{Z}$, the defining relation is*

$$\sum_{l=0}^\infty f_l T_{n-l}T_{m+l} - \sum_{l=0}^\infty f_l T_{n-l}T_{m+l} = -\frac{(1-q_1)(1-q_3)}{1-q_2^{-1}}\left(q_2^{-n}-q_2^n\right)\delta_{n+m,0}. \tag{0.0.56}$$

Denote $T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n}$, $\delta(x) = \sum_{k\in\mathbb{Z}} x^k$. Relation (0.0.56) *is equivalent to*

$$f(w/z)T(z)T(w) - f(z/w)T(w)T(z) = -\frac{(1-q_1)(1-q_3)}{1-q_2^{-1}}\left(\delta\left(\frac{w}{q_2z}\right) - \delta\left(\frac{q_2w}{z}\right)\right). \tag{0.0.57}$$

Denote $\beta(x) = (vx;v^4)_\infty/(v^3x;v^4)_\infty$.

Theorem 0.0.6 (Theorem 2.4.1)**.** *The following formula determine action of Vir_{q_1,q_2} on V_j for any $j\in\mathbb{Z}$.*

$$T(z) = z^{1/2}\frac{v^{3/2}(1-q_1)}{\beta(vq_1)}\Big(u\Psi^*_+(vz)\Phi_+(q_1z) + u^{-1}\Psi^*_-(vz)\Phi_-(q_1z)\Big). \tag{0.0.58}$$

Twisted deformed Virasoro algebra

Definition 0.0.5. *Twsited deformed Virasoro algebra $\mathsf{Vir}^{tw}_{q_1,q_2}$ is generated by T_r for $r \in 1/2+\mathbb{Z}$, the defining relation is*

$$\sum_{l=0}^{\infty} f_l T_{r-l} T_{s+l} - \sum_{l=0}^{\infty} f_l T_{s-l} T_{r+l} = -\frac{(1-q_1)(1-q_3)}{1-q_2^{-1}} \left(q_2^{-r} - q_2^{r}\right) \delta_{r+s,0}. \qquad (0.0.59)$$

Denote $T(z) = \sum_{r\in 1/2+\mathbb{Z}} T_r z^{-r}$, $\delta_{\mathrm{odd}}(x) = \sum_{r\in 1/2+\mathbb{Z}} x^r$ Relation (0.0.59) is equivalent to

$$f(w/z)T(z)T(w) - f(z/w)T(w)T(z) = -\frac{(1-q_1)(1-q_3)}{1-q_2^{-1}} \left(\delta_{\mathrm{odd}}\left(\frac{w}{q_2 z}\right) - \delta_{\mathrm{odd}}\left(\frac{q_2 w}{z}\right)\right). \qquad (0.0.60)$$

Theorem 0.0.7 (Theorem 2.4.2)**.** *The following formula determine action of $\mathsf{Vir}^{tw}_{q_1,q_2}$ on F_i for $i = 0, 1$*

$$T(z) = (-1)^{1/2} \frac{v^{3/2}(q_1^{-1/2} - q_1^{1/2})}{\beta(vq_1)} \left(q_1 z \Psi_-^*(vz)\Phi_+(q_1 z) + \Psi_+^*(vz)\Phi_-(q_1 z)\right). \qquad (0.0.61)$$

Twisted Cherednik representation

Double affine Hecke algebra

Below we will recall a definition and basic properties of double affine Hecke algebra (DAHA) [Che92, Kir97, Che05].

Definition 0.0.6. *Double affine Hecke algebra $\mathcal{H}_N$, is an algebra with generators $T_1, \ldots, T_{N-1}$, $\pi^{\pm 1}$, $Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}$ and relations* [1]

$$(T_i - v)(T_i + v^{-1}) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad (0.0.62)$$

$$T_i Y_i T_i = Y_{i+1}, \quad T_i Y_j = Y_j T_i, j \neq i, i+1 \qquad (0.0.63)$$

$$\pi Y_i \pi^{-1} = q^{\delta_{i,N}} Y_{i+1}, \quad Y_i Y_j = Y_j Y_i \qquad (0.0.64)$$

$$\pi T_i \pi^{-1} = T_{i+1}, \quad \pi^N T_i = T_i \pi^N \qquad (0.0.65)$$

Here and below we use the notation $Y_1 = Y_{N+1}$.

The operators $T_1, \ldots, T_{N-1}$ generated finite Hecke algebra H. The operators $T_1, \ldots, T_{N-1}, Y_1, \ldots, Y_N$ generated affine Hecke algebra H^Y. The operators $T_1, \ldots, T_{N-1}, \pi$ generated affine Hecke algebra H^X; one can define

$$X_i = T_i \ldots T_{N-1} \pi^{-1} T_1^{-1} \ldots T_{i-1}^{-1}. \qquad (0.0.66)$$

Let $\widetilde{SL}(2,\mathbb{Z})$ be the braid group on three stands. More precisely, $\widetilde{SL}(2,\mathbb{Z})$ is generated by τ_+ and τ_- with the relation $\tau_+\tau_-^{-1}\tau_+ = \tau_-^{-1}\tau_+\tau_-^{-1}$. The reason for our notation is that $\widetilde{SL}(2,\mathbb{Z})$ is a central extension of $SL(2,\mathbb{Z})$ by $\mathbb{Z}$, the projection is given by

$$p(\tau_+) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad p(\tau_-) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad (0.0.67)$$

The kernel of the projection p is generated by $(\tau_+\tau_-^{-1}\tau_+)^4$.

Proposition 0.0.5 ([Che05])**.** *There is an action of $\widetilde{SL}(2,\mathbb{Z})$ on $\mathcal{H}_N$ determined by the following formulas*

$$\tau_+: \quad T_i \to T_i, \;\; X_i \to X_i, \;\; Y_i \to Y_i X_i T_{i-1} \ldots T_1 T_1 \ldots T_{i-1} \qquad (0.0.68)$$

$$\tau_-: \quad T_i \to T_i, \;\; X_i \to X_i Y_i T_{i-1}^{-1} \ldots T_1^{-1} T_1^{-1} \ldots T_{i-1}^{-1}, \;\; Y_i \to Y_i \qquad (0.0.69)$$

[1] We were using parameter q in Sections *Schur specialization case* and *Twisted q-W-algebra*. Parameter q here and below has a different meaning. We hope that, this will not lead to a confusion.

Cherednik representation

Algebra H^X has one dimensional representation $\mathbb{C}_u$

$$T_i \mapsto v, \quad \pi \mapsto u. \tag{0.0.70}$$

Cherednik representation $\mathbf{C}_u$ of $\mathcal{H}_N$ is the induced of representation $\mathbb{C}_u$. Representation $\mathbf{C}_u$ can be identified with Laurent polynomials $\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}]$. Action of T_i and π is given by the following formulas

$$T_i = s_i^Y + (v - v^{-1})\frac{s_i^Y - 1}{Y_i/Y_{i+1} - 1}, \qquad \pi(Y_1^{\lambda_1} Y_2^{\lambda_2} \ldots Y_N^{\lambda_N}) = uq^{\lambda_N} Y_1^{\lambda_N} Y_2^{\lambda_1} Y_3^{\lambda_2} \ldots Y_N^{\lambda_{N-1}},$$

here s_i^Y is the transposition of Y_i and Y_{i+1}.

Twisted Cherednik representation

In this Section we will give an explicit construction of twisted Cherednik representation.

Action of affine Hecke algebra Let $\mathbb{C}^n$ be a vector space with the basis $e_0, \ldots, e_{n-1}$. Define R-matrix acting on $\mathbb{C}^n \otimes \mathbb{C}^n$

$$R = \sum_a vE_{aa} \otimes E_{a,a} + \sum_{a<b} \Big(E_{ab} \otimes E_{ba} + E_{ba} \otimes E_{ab} + (v - v^{-1})E_{aa} \otimes E_{bb} \Big).$$

Define action of Hecke algebra H on $(\mathbb{C}^n)^{\otimes N}$ by the formula $T_i \mapsto R_{i,i+1}$, here the indices encodes tensor factors on which R-acts. We can consider the induced representation of H^Y on $(\mathbb{C}^n)^{\otimes N}[Y_1^{\pm 1}, \ldots Y_N^{\pm 1}]$.

We can write action of T_i explicitly. Let s_i^Y be the operators on $(\mathbb{C}^n)^{\otimes N}[Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}]$ which swaps Y_i and Y_{i+1}. Let s_i^e be the operator on $(\mathbb{C}^n)^{\otimes N}[Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}]$ which swaps tensor factors number i and $i+1$ (and commutes with all Y_j). Finally, let $s_i = s_i^Y s_i^e$. The action of T_i is given by the following formula

$$T_i = s_i^Y R_{i,i+1} + (v - v^{-1})\frac{s_i^Y - 1}{Y_i/Y_{i+1} - 1} \tag{0.0.71}$$

The obtained representation of affine Hecke algebra H^Y is well-know. It appears in the context of quantum affine Shcur-Weyl duality [GRV94], [CP94].

Action of DAHA Below we will use identification

$$\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \to (\mathbb{C}^n)^{\otimes N} [Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}] \tag{0.0.72}$$

$$(Y^{j_1} e_{i_1}) \otimes \cdots \otimes (Y^{j_N} e_{i_N}) \mapsto Y_1^{j_1} \ldots Y_n^{j_n} e_{i_1} \otimes \cdots \otimes e_n \tag{0.0.73}$$

Let us define $e_i \in \mathbb{C}^n[Y^{\pm 1}]$ for $i \in \mathbb{Z}$ by

$$e_i = Y^{-1} e_{i+n}. \tag{0.0.74}$$

Let us define operators κ and D acting on $\mathbb{C}^n[Y^{\pm 1}]$ by the formula $\kappa e_i = e_{i-1}$, $D(Y^j e_a) = u_a q^j Y^j e_a$ for $a = 0, \ldots, n-1$. By κ_i and D_i we denote the corresponding operators acting by κ and D on i-th tensor factor.

Theorem 0.0.8 (Theorem 3.2.1)**.** *For any $n_{tw} \in \mathbb{Z}$, there is an action of algebra $\mathcal{H}_N$ on $(\mathbb{C}^n)^{\otimes N}[Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}]$ determined as follows*

- *subalgebra H^Y acts as discribed above*
- $\pi = \kappa_1^{n_{tw}} D_1 s_1 \ldots s_{N-1}$

Denote the obtained representation by $\mathbf{C}_{u_0,\ldots,u_{n-1}}^{(n_{tw},n)}$.

Isomorphism with twisted Cherednik representation Twisted Cherednik representation $\mathbf{C}_u^\tau$ is determined by the projection $p(\tau) \in SL(2,\mathbb{Z})$, since $(\tau_+\tau_-^{-1}\tau_+)^4$ acts by an exterior automorphism of $\mathcal{H}_N$.

Theorem 0.0.9 (Theorem 3.2.4)**.** *Let $n_{tw} = n'$ be mutually prime with n, then $\mathbf{C}_{u_0,\ldots,u_{n-1}}^{(n_{tw},n)}$ is isomorphic to twisted Cherednik representation $\mathbf{C}_u^\tau$ for*

$$p(\tau) = \begin{pmatrix} m' & m \\ n' & n \end{pmatrix} \qquad u = u_0 \ldots u_{n-1} q^{1-n} \tag{0.0.75}$$

We can note that the representation $\mathbf{C}_{u_0,\ldots,u_{n-1}}^{(n',n)}$ is determined (up to an isomorphism) by the product $u_0 \ldots u_{n-1}$. For example, we can choose $u_i = u^{\frac{1}{n}} q^{\frac{n+2i-1}{2n}}$. Denote

$$\mathbf{C}_u^{(n',n)} = \mathbf{C}_{u_0,\ldots,u_{n-1}}^{(n',n)} \qquad \text{for } u_i = u^{\frac{1}{n}} q^{\frac{n+2i-1}{2n}} \tag{0.0.76}$$

Twisted Fock module

Toroidal algebra

Toroidal algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ depends on two parameters q_1 and q_2. Let us also introduce parameter q_3 such that $q_1 q_2 q_3 = 1$. The algebra has a presentation via generators $P_{a,b}$ for $(a,b) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ and central elements c, c'. We will not need explicit form of the relations. Though these presentation is used to obtain the following result.

Proposition 0.0.6 ([SV13b])**.** *Group $\widetilde{SL}(2,\mathbb{Z})$ acts on $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ via automorphism.*

Let us consider an element $\tau \in \widetilde{SL}(2,\mathbb{Z})$ such that the projection is

$$p(\tau) = \begin{pmatrix} m' & m \\ n' & n \end{pmatrix} \tag{0.0.77}$$

Then action of τ is given by the following formulas

$$\tau(c) = c^n \left(c'\right)^m \qquad \tau(c') = c^{n'} \left(c'\right)^{m'} \tag{0.0.78}$$

$$\tau\left(P_{a,b}\right) = c^{n_\tau(a,b)} \left(c'\right)^{m_\tau(a,b)} P_{m'a+mb,n'a+nb}, \tag{0.0.79}$$

here $n_\tau(a,b)$ and $m_\tau(a,b)$ are certain integers, determined by τ and (a,b).

Chevalley generators The algebra has another presentation. The generators are $P_{1,b}$, $P_{0,b}$, $P_{-1,b}$, and central elements c, c'. To describe the relations let us define

$$E(z) = \sum_{b\in\mathbb{Z}} P_{1,b} z^{-b} \qquad F(z) = \sum_{b\in\mathbb{Z}} P_{-1,b} z^{-b} \tag{0.0.80}$$

Define

$$\sum_{k>0} \theta_{\pm k} z^{-k} = \exp\left(\sum_k \frac{(1-q_2^k)(1-q_3^k)}{k} P_{0,\pm k} z^{-k}\right) \tag{0.0.81}$$

For $k \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}$

$$[P_{0,k}, P_{0,-k}] = k\frac{(1-q_1^k)(c^k - c^{-k})}{(1-q_2^k)(1-q_3^k)} \tag{0.0.82}$$

$$[P_{0,k}, P_{1,b}] = c^{-k}(q_1^k - 1) P_{1,b+k} \qquad [P_{0,-k}, P_{1,b}] = (1-q_1^k) P_{1,b-k} \tag{0.0.83}$$

$$[P_{0,k}, P_{-1,b}] = (1-q_1^k) P_{1,b+k} \qquad [P_{0,-k}, P_{-1,b}] = (q_1^k - 1) c^k P_{1,b-k} \tag{0.0.84}$$

$$(z-q_1w)(z-q_2w)(z-q_3w)E(z)E(w) = -(w-q_1z)(w-q_2z)(w-q_3z)E(w)E(z) \tag{0.0.85}$$
$$(z-q_1w)(z-q_2w)(z-q_3w)F(z)F(w) = -(w-q_1z)(w-q_2z)(w-q_3z)F(w)F(z) \tag{0.0.86}$$

For $a+b>0$

$$[P_{1,a}, P_{-1,b}] = \frac{(1-q_1)c^a c'}{(1-q_2)(1-q_3)}\theta_{a+b} \qquad [P_{1,-a}, P_{-1,-b}] = -\frac{(1-q_1)c^{-b}(c')^{-1}}{(1-q_2)(1-q_3)}\theta_{-a-b} \tag{0.0.87}$$

For $a \in \mathbb{Z}$

$$[P_{1,a}, P_{-1,-a}] = \frac{(1-q_1)\left(c^a c' - c^{-a}(c')^{-1}\right)}{(1-q_2)(1-q_3)} \tag{0.0.88}$$
$$[P_{1,a}, [P_{1,a-1}, P_{1,a+1}]] = 0 \tag{0.0.89}$$
$$[P_{-1,a}, [P_{-1,a-1}, P_{-1,a+1}]] = 0 \tag{0.0.90}$$

Fock representation

Recall notation for Heisenberg algebra $[\mathbf{a}_k, \mathbf{a}_l] = k\delta_{k+l,0}$. Also recall that Fock representation is denoted by $F^{\mathbf{a}}$.

Proposition 0.0.7 ([FHH+09])**.** *The following formulas determine an action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ on $F^{\mathbf{a}}$.*

$$c \mapsto v^{-1}, \qquad c' \mapsto 1, \tag{0.0.91}$$
$$P_{0,-j} \mapsto q_1^j \mathbf{a}_{-j}, \qquad P_{0,j} \mapsto \frac{q_1^j - 1}{q_3^{-j}-1} v^{-jn}\mathbf{a}_j, \tag{0.0.92}$$
$$E(z) = \frac{q_1 u}{1-q_3}\exp\left(-\sum_k \frac{q_1^k - q_2^{-k}}{k}\mathbf{a}_{-k}z^k\right)\exp\left(\sum_k \frac{1-q_1^{-k}}{k}\mathbf{a}_k z^{-k}\right) \tag{0.0.93}$$
$$F(z) = \frac{u^{-1}}{1-q_3^{-1}}\exp\left(\sum_k \frac{v^k - q_1^k v^{-k}}{k}\mathbf{a}_{-k}z^k\right)\exp\left(\sum_k \frac{v^{-k}(q_1^{-k}-1)}{k}\mathbf{a}_k z^{-k}\right) \tag{0.0.94}$$

The representation obtained is called *Fock module* and will be denoted $\mathcal{F}_u$.

Remark 0.0.1. Limit $q_2 \to 1$ of representation $\mathcal{F}_u$ is isomorphic to $\mathscr{F}^{(0,-1)}_{-q^{3/2}u^{-1}}$.

Bosonization of $U_v(\widehat{\mathfrak{gl}}_n)$ and its vertex operators

The algebra $U_v(\widehat{\mathfrak{gl}}_n)$ has n integrable representations of level 1. Denote them by $F_0, \dots, F_{n-1}$. There are vertex operators

$$\Phi^{(i-1,i)}(z)\colon F_i \to F_{i-1}\otimes V_z, \qquad \Psi^{(i-1,i)}(z)\colon F_i \to V_z \otimes F_{i-1}, \tag{0.0.95}$$
$$\Phi^{*(i+1,i)}(z)\colon F_i \otimes V_z \to F_{i+1}, \qquad \Psi^{*(i+1,i)}(z)\colon V_z\otimes F_i \to F_{i+1}. \tag{0.0.96}$$

Here V_z denotes n-dimensional evaluation representation with the standard basis $e_0, \dots, e_{n-1}$. Let us introduce operators $\Phi_0(z), \dots, \Phi_{n-1}(z)$ and $\Phi_0^*(z), \dots, \Phi_{n-1}^*(z)$

$$\Phi(z)\mathbf{w} = \sum_{a=0}^{n-1}\Phi_a(z)\mathbf{w}\otimes e_a \qquad \Phi^*(z)(\mathbf{w}\otimes e_a) = \Phi_a(z)\mathbf{w} \tag{0.0.97}$$

Analogously, one can define $\Psi_0(z), \dots, \Psi_{n-1}(z)$ and $\Psi_0^*(z), \dots, \Psi_{n-1}^*(z)$.

Realizations of twisted Fock representations

Recall that $v = q_2^{-1/2}$. There is an isomorphism $U_v(\widehat{\mathfrak{gl}}_n) = U_v(\widehat{\mathfrak{sl}}_n)\otimes U_v(\mathsf{Heis})$. Let B_j be the generators of $U_v(\mathsf{Heis})$, satisfying the relation $[B_k, B_l] = k[n]^+_{v^k}\delta_{k+l,0}$.

Theorem 0.0.10 (Theorem 3.6.1). *The following formulas determine an action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ on F_i*

$$c \mapsto v^{-n}, \qquad c' \mapsto v^{-n'}, \qquad P_{0,-j} \mapsto q_1^j B_{-j}, \qquad P_{0,j} \mapsto \frac{q_1^j - 1}{q_3^{-j}-1} v^{-jn} B_j, \tag{0.0.98}$$

$$E(z) \mapsto u^{-\frac{1}{n}} q^{\frac{n+1}{2n}} \sum_{\alpha-\beta\equiv n' \bmod n} q_1^{-\frac{\alpha}{n}} z^{\frac{\beta-\alpha+n'}{n}} \Phi^*_{(\alpha)}(q_1 z)\Psi_{(\beta)}(z), \tag{0.0.99}$$

$$F(z) \mapsto u^{\frac{1}{n}} q^{\frac{n-1}{2n}} v^{-n'} \sum_{\alpha-\beta\equiv -n' \bmod n} q_1^{\frac{\beta}{n}} z^{\frac{\beta-\alpha-n'}{n}} \Psi^*_{(\alpha)}(v^n z)\Phi_{(\beta)}(q_1 v^n z). \tag{0.0.100}$$

The representation obtained is isomorphic to $\mathcal{F}^\sigma_{u'}$ for $u' = uv^{-n_{\sigma^{-1}}(1,0)}$.

Remark 0.0.2. The formulas (0.0.61) and close to (0.0.100) but has different shifts of the arguments and coefficients. The differences are yield by different normalization for vertex operators of $U_v(\widehat{\mathfrak{sl}}_2)$ and $U_v(\widehat{\mathfrak{gl}}_n)$.

Semi-infinite construction

Connection of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ and spherical DAHA Denote

$$[i]^\pm_v = \frac{v^{\pm 2i}-1}{v^{\pm 2}-1} \qquad [k]_v = \frac{v^i - v^{-i}}{v - v^{-1}} \tag{0.0.101}$$

$$[k]!^\pm_v = [1]^\pm_v \dots [k]^\pm_v \qquad [k]!_v = [1]_v \dots [k]_v \tag{0.0.102}$$

In this Section we will consider $\mathcal{H}_N$ for different parameters q and v, therefore we will write $\mathcal{H}_N(q,v)$. Let $\mathbf{S}_+$ and $\mathbf{S}_-$ be symmetrizer and antisymmetrizer.

$$\mathbf{S}_+ = \frac{1}{[N]!^+_v} \sum v^{l(\sigma)} T_\sigma \qquad \mathbf{S}_- = \frac{1}{[N]!^-_v} \sum (-v)^{-l(\sigma)} T_\sigma \tag{0.0.103}$$

Let $S_\pm\mathcal{H}_N(q,v) = \mathbf{S}_\pm\mathcal{H}_N(q,v)\mathbf{S}_\pm$ be the corresponding spherical subalgebras of DAHA.

Theorem 0.0.11 ([SV11]). *There is a surjection from $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ onto $S_+\mathcal{H}_N(q,v)$ for $q_1 = q$, $q_2 = v^2$.*

Denote the image of $P_{a,b}$ by $P^{(N)}_{a,b}$.

Proposition 0.0.8 ([SV11]). *The following formulas hold for $k > 0$ and $b \in \mathbb{Z}$*

$$P^{(N)}_{0,k} = \mathbf{S}_+(Y_1^k + \dots + Y_N^k)\mathbf{S}_+ \qquad P_{0,-k} = q^k \mathbf{S}_+(Y_1^{-k} + \dots + Y_N^{-k})\mathbf{S}_+ \tag{0.0.104}$$

$$P^{(N)}_{1,b} = q[N]^-_v \mathbf{S}_+ X_1 Y_1^b \mathbf{S}_+ \qquad P^{(N)}_{-1,b} = [N]^+_v \mathbf{S}_+ Y_1^b X_1^{-1} \mathbf{S}_+ \tag{0.0.105}$$

Proposition 0.0.9. *There is an algebra isomorphism $S_-\mathcal{H}_N(q,v) \cong S_+\mathcal{H}_N(q,-v^{-1})$.*

Corollary 0.0.10. *There is a surjection from $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ onto $S_-\mathcal{H}_N(q,v)$ for $q_1 = q$, $q_2 = v^{-2}$. Moreover*

$$P^{(N)}_{0,k} = \mathbf{S}_-(Y_1^k + \dots + Y_N^k)\mathbf{S}_- \qquad P_{0,-k} = q^k \mathbf{S}_-(Y_1^{-k} + \dots + Y_N^{-k})\mathbf{S}_- \tag{0.0.106}$$

$$P^{(N)}_{1,b} = q[N]^+_v \mathbf{S}_- X_1 Y_1^b \mathbf{S}_- \qquad P^{(N)}_{-1,b} = [N]^-_v \mathbf{S}_- Y_1^b X_1^{-1} \mathbf{S}_- \tag{0.0.107}$$

The limit Recall that as vector space $\mathbf{C}_u^{(n',n)}$ is identified with $\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$. Consider an inductive system

$$\mathbf{S}_-\mathbb{C}^n[Y^{\pm 1}] \xrightarrow{\varphi_{2,1}^{(m)}} \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes 2} \xrightarrow{\varphi_{3,2}^{(m)}} \dots \xrightarrow{\varphi_{N,N-1}^{(m)}} \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \xrightarrow{\varphi_{N+1,N}^{(m)}} \dots \tag{0.0.108}$$

with the maps $\varphi_{N+1,N}^{(m)}(w) = \mathbf{S}_-\left(w \otimes e_{N-m}\right)$. Denote the inductive limit by $\Lambda_{v,m}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$. Also let us denote $\varphi_{R,N}^{(m)}(w) = \mathbf{S}_-\left(w \otimes e_{N-m} \otimes e_{N+1-m} \otimes \dots \otimes e_{R-1-m}\right)$.

Definition 0.0.7. *Action of operators* $A^{(N)} \colon \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \to \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$ stabilizes *if there is* N_0 *such that for any* $N > N_0$ *we have*

$$\varphi_{N+n,N}^{(m)} \circ A^{(N)} = A^{(N+n)} \circ \varphi_{N+n,N}^{(m)} \tag{0.0.109}$$

Note that if $A^{(N)}$ stabilizes, then there are induced operators $\hat{A}$ on $\Lambda_{v,m}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$.

Below we will use notation $q_1 = q$, $q_2 = v^{-2}$ and $P_{a,b}^{(N)}$, cf. Corollary 0.0.10

Proposition 0.0.11. *Operators* $v^{\frac{aN}{n}} P_{a,b}^{(N)}$ *stabilizes for* $an' + bn < 0$.

Definition 0.0.8. *Action of operators* $A^{(N)} \colon \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \to \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$ converges *if for any* $w \in \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$ *the* $R \to \infty$

$$\varphi_{N+nR}^{(m)} \circ A^{(N+nR)} \circ \varphi_{N+nR,N}^{(m)}(w) \tag{0.0.110}$$

Remark 0.0.3. Note that $\Lambda_{v,m}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$ is a graded vector space with finite dimensional graded components. Therefore convergence of sequence (0.0.110) is understood in sence of finite dimensional vector spaces over $\mathbb{C}$.

Proposition 0.0.12. *Operators* $v^{\frac{aN}{n}} P_{a,b}^{(N)}$ *converge for* $a \geq 0$ *and* $|q_3| < 1$.

Proposition 0.0.13. *Operators* $v^{\frac{aN}{n}} P_{a,b}^{(N)}$ *converge for* $a \leq 0$ *and* $|q_3| > 1$.

Moreover, these operators admit an analytic continuation and can be defined for $q_3 \neq 1$. Denote the obtained operators by $\hat{P}_{a,b}$.

Theorem 0.0.12. *The following formulas determine an action of* $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ *on* $\Lambda_{v,i}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$

$$c \mapsto v^{-n} \qquad c' \mapsto v^{-n'} \tag{0.0.111}$$

$$P_{0,-j} \mapsto q^j \hat{P}_{0,-j} \qquad P_{0,j} \mapsto \frac{q_1^j - 1}{v^{-2j} q_1^j - 1} v^{-jn} \hat{P}_{0,j} \tag{0.0.112}$$

$$P_{1,b} \mapsto \hat{P}_{1,b} \qquad P_{-1,b} \mapsto v^{-nb-n'} \hat{P}_{1,b} \tag{0.0.113}$$

The representation obtained is isomorphic to the representation from Theorem 0.0.10.

Remark 0.0.4. Technically, we prove Theorems 0.0.10 and 0.0.12 simultaneously . There is a natural identification $F_i = \Lambda_{v,i}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$ [KMS95, LT00]. Using the identification it is not hard to see that the formulas coincide. Therefore we can use either of the formulas to check the relations of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$.

Our principal method is semi-infinite construction. The operators $P_{a,b}$ for $an' + bn < 0$ satisfy the relations since the operators stabilizes. Verification of the other relations $\mathcal{F}_u^{\sigma}$. The proof contains both arguments via limit and explicit calculation.

Chapter 1

Schur specialization case

1.1 Introduction

Toroidal algebra Representation theory of quantum toroidal algebras has been actively developed in recent years. This theory has numerous applications, including geometric representation theory and AGT relation [Neg18], topological strings [AFS12], integrable systems, knot theory [GN15], and combinatorics [CM18].

In this thesis we consider only the quantum toroidal $\mathfrak{gl}_1$ algebra; we denote it by $U_{q,t}(\ddot{\mathfrak{gl}}_1)$. The algebra depends on two parameters q, t and has PBW generators $E_{k,l}$, $(k,l) \in \mathbb{Z}^2$ and central generators c',c [BS12a]. In the main part of the text we consider only the case $q = t$, where toroidal algebra becomes the universal enveloping of the Lie algebra with these generators $E_{k,l}$, c', c and the relation

$$[E_{k,l}, E_{r,s}] = (q^{(sk-lr)/2} - q^{(lr-sk)/2})E_{k+r,l+s} + \delta_{k,-r}\,\delta_{l,-s}(c'k + cl). \tag{1.1.1}$$

We denote this Lie algebra by $\mathfrak{Diff}_q$, since there is a homomorphism from this algebra to the algebra of q-difference operators generated by D, x with the relation $Dx = qxD$; namely $E_{k,l} \mapsto q^{kl/2}x^l D^k$.

There is another presentation of the algebra $\mathfrak{Diff}_q$ (and more generally $U_{q,t}(\ddot{\mathfrak{gl}}_1)$) using the Chevalley generators $E(z) = \sum_{k\in\mathbb{Z}} E_{1,k}z^{-k}$, $F(z) = \sum_{k\in\mathbb{Z}} E_{-1,k}z^{-k}$, $H(z) = \sum_{k\neq 0} E_{0,k}z^{-k}$, see e.g. [Tsy17].

In this thesis we deal with the Fock representations of $\mathfrak{Diff}_q$; to be more precise there is a family $\mathcal{F}_u$ of Fock modules, depending on the parameter u (see Proposition 1.3.1 for a construction of $\mathcal{F}_u$). They are just Fock representations of the Heisenberg algebra generated by $E_{0,k}$. The images of $E(z)$ and $F(z)$ are vertex operators. A construction of this type is usually called *bosonization*.

It was shown in [FHH$^+$09], [Neg18] that the image of toroidal algebra $U_{q,t}(\ddot{\mathfrak{gl}}_1)$ in the endomorphisms of the tensor product of n Fock modules is the deformed W-algebra for $\mathfrak{gl}_n$. There is the so-called conformal limit $q, t \to 1$, in which deformed W-algebras go to vertex algebras. These vertex algebras are tensor products of the Heisenberg algebra and the W-algebras of $\mathfrak{sl}_n$. In the case $q = t$, the central charge of the corresponding W-algebra of $\mathfrak{sl}_n$ is equal to $n - 1$. These W-algebras appear in the study of isomonodromy/CFT correspondence (see [GIL12], [GM16]). This is one of the motivations of this chapter.

The q-deformation of the isomonodromy/CFT correspondence was proposed in [BS17b], [BGT19], [JNS17]. The main statement is an explicit formula for the q-isomonodromic tau function as an infinite sum of conformal blocks for deformed certain W-algebras with $q = t$. In general, these tau functions are complicated, but there are special cases (corresponding to algebraic solutions) where these tau functions are very simple ([BS17b], [BGM19]). These cases should correspond to special representations of q-deformed W-algebras. The construction of such representation is one of the purposes of this chapter.

Twisted Fock modules There is a natural action of $SL(2,\mathbb{Z})$ on $\mathfrak{Diff}_q$. We will parametrize $\sigma \in SL(2,\mathbb{Z})$ by

$$\sigma = \begin{pmatrix} m' & m \\ n' & n \end{pmatrix}. \tag{1.1.2}$$

Then σ acts as

$$\sigma(E_{k,l}) = E_{m'k+ml,\, n'k+nl}, \quad \sigma(c') = m'c' + n'c, \quad \sigma(c) = mc' + nc. \tag{1.1.3}$$

For any $\mathfrak{Diff}_q$ module M and $\sigma \in SL(2,\mathbb{Z})$, we denote by M^σ the module twisted by the automorphism σ (see Definition 1.2.3). The twisted Fock modules depend only on n and n' (up to isomorphism). These numbers are the values of the central generators c and c', correspondingly, acting on $\mathcal{F}_u^\sigma$. Therefore we will also use the notation $\mathcal{F}_u^{(n',n)}$ for $\mathcal{F}_u^\sigma$. Twisted Fock modules $\mathcal{F}_u^\sigma$ (for generic q,t) were used, for example, in [AFS12] and [GN17].

In Section 1.4 we construct explicit bosonization of the twisted Fock modules $\mathcal{F}_u^\sigma$ for $q = t$. Actually, we give three constructions: the first one in terms of n-fermions (see Theorem 1.4.1), the second one in terms of n-bosons (see Theorem 1.4.2) and the third one in terms of one twisted boson (see Theorem 1.4.3) (here, for simplicity, we assume that $n > 0$). In other words, any twisted Fock module will be identified with the basic module for $\widehat{\mathfrak{gl}}_n$; these two bosonizations correspond to homogeneous [FK81] and principal [LW78],[KKLW81] constructions.

The construction of the bosonization is nontrivial, because it is given in terms of Chevalley generators (note that the $SL(2,\mathbb{Z})$ action is not easy to describe in terms of Chevalley generators). The appearance of affine $\mathfrak{gl}_n$ is in agreement with the Gorsky-Neguț conjecture [GN17]. More specifically, it was conjectured in [GN17] that there exists an action (with certain properties) of $U_{p^{1/2}}(\widehat{\mathfrak{gl}}_n)$ on $\mathcal{F}_u^\sigma$ for $p = q/t \neq 1$; we expect this to be p-deformation of the $\widehat{\mathfrak{gl}}_n$-action constructed in this chapter.

It is instructive to look at the formulas in the simplest examples. For simplicity, we give here only formulas for $E(z)$. Here we introduce the notation in a sloppy way (for details see Sections 1.3 and 1.4).

Example 1.1.1. In the standard case $n = 1$, $n' = 0$ we have

$$E(z) = uq^{-1/2} z\psi(q^{-1/2}z)\psi^*(q^{1/2}z) = \frac{u}{1-q} :\exp\left(\phi(q^{1/2}z) - \phi(q^{-1/2}z)\right):, \tag{1.1.4}$$

where $\psi(z), \psi^*(z)$ are complex conjugate fermions (see Section 1.3.2), $\phi(z) = \sum_{j\neq 0} a[j]z^{-j}/j$ is a boson and $a[j]$ are generators of the Heisenberg algebra with relation $[a[j], a[j']] = j\delta_{j+j',0}$ (see Section 1.3.1).

Example 1.1.2. The first nontrivial case is given by $n = 2$, $n' = 1$. We have three formulas (corresponding to Theorems 1.4.1, 1.4.2, 1.4.3):

$$E(z) = u^{\frac12} q^{-\frac14}\left(z^2\psi_{(0)}(q^{-1/2}z)\psi^*_{(1)}(q^{1/2}z) + z\psi_{(1)}(q^{-1/2}z)\psi^*_{(0)}(q^{1/2}z)\right), \tag{1.1.5}$$

$$E(z) = u^{\frac12} q^{-\frac14}\left(z^2 :\exp\left(\phi_1(q^{1/2}z) - \phi_0(q^{-1/2}z)\right): + z :\exp\left(\phi_0(q^{1/2}z) - \phi_1(q^{-1/2}z)\right):\right)(-1)^{a_0[0]}, \tag{1.1.6}$$

$$E(z) = \frac{z^{\frac12}u^{\frac12}}{2(1-q^{\frac12})}\left(:\exp\left(\sum_{k\neq 0} \frac{q^{-k/4} - q^{k/4}}{k} a_k z^{-k/2}\right): - :\exp\left(\sum_{k\neq 0} (-1)^k \frac{q^{-k/4} - q^{k/4}}{k} a_k z^{-k/2}\right):\right). \tag{1.1.7}$$

Here $\psi_{(0)}(z), \psi^*_{(0)}(z)$ and $\psi_{(1)}(z), \psi^*_{(1)}(z)$ are anticommuting pairs of complex conjugate fermions (see Section 1.4.1), $\phi_b(z) = \sum_{j\neq 0} a_b[j]z^{-j}/j + Q + a_b[0]\log z$ are commuting bosons, and $a_b[j]$ are generators of the Heisenberg algebra with the relation $[a_b[j], a_{b'}[j']] = j\delta_{j+j',0}\delta_{b,b'}$ (see Section 1.4.2). The generators a_k in (1.1.7) satisfy $[a_k, a_{k'}] = k\delta_{k+k',0}$.

The relation between (1.1.5) and (1.1.6) is a standard boson-fermion correspondence. In the right-hand side of formula (1.1.7) we have only one Heisenberg algebra with generators a_k, but since we have both integer and half-integer powers of z, one can think that we have a boson with a nontrivial monodromy. This is the reason for the term 'twisted boson'; we will also call this construction *strange bosonization*. Note that half-integer powers of z cancel in the right-side of (1.1.7).

We present two different proofs of Theorems 1.4.1, 1.4.2, and 1.4.3. The first one is given in Section 1.5 and is based on the following idea. For any full rank sublattice $\Lambda \in \mathbb{Z}^2$ of index n, we have a subalgebra $\mathfrak{Diff}^{\Lambda}_{q^{1/n}} \subset \mathfrak{Diff}_{q^{1/n}}$, which is spanned by $E_{a,b}$ for $(a,b) \in \Lambda$ and central elements c, c'. The algebra $\mathfrak{Diff}^{\Lambda}_{q^{1/n}}$ is isomorphic to $\mathfrak{Diff}_q$; the isomorphism depends on the choice of a positively oriented basis v_1, v_2 in Λ. Denote this isomorphism by ϕ_{v_1,v_2}.

If the basis v_1, v_2 is such that $v_1 = (N, 0)$, $v_2 = (R, d)$, then the restriction of the Fock module $\mathcal{F}_u$ on $\phi_{v_1,v_2}(\mathfrak{Diff}_q)$ is isomorphic to the sum of tensor products of the Fock modules

$$\mathcal{F}_{u^{1/N}}\big|_{\phi_{v_1,v_2}(\mathfrak{Diff}_q)} \cong \bigoplus_{l \in \mathbf{Q}_{(d)}} \mathcal{F}_{uq^{rl_0}} \otimes \cdots \otimes \mathcal{F}_{uq^{r(\frac{\alpha}{n}+l_\alpha)}} \otimes \cdots \otimes \mathcal{F}_{uq^{r\left(\frac{d-1}{d}+l_{d-1}\right)}} \tag{1.1.8}$$

where $r = \gcd(N, R)$ and $\mathbf{Q}_{(d)} = \{(l_0, \ldots, l_{d-1}) \in \mathbb{Z}^d | \sum l_i = 0\}$. If we choose basis w_1, w_2 in Λ which differs from v_1, v_2 by $\sigma \in SL(2, \mathbb{Z})$, we get an analogue of decomposition (1.1.8) with RHS given by a sum of tensor products of the twisted Fock modules. For the basis $w_1 = (r, n_{tw})$, $w_2 = (0, n)$, we write formulas for Chevalley generators of $\mathfrak{Diff}_q = \mathfrak{Diff}^{\Lambda}_{q^{1/n}}$ using either initial fermion or initial boson for $\mathcal{F}_u$. Applying this for the lattices with $d = 1$, we get Theorems 1.4.1, 1.4.2, 1.4.3.

The second proof of these theorems is based on the semi-infinite construction. Let V_u denote the representation of the algebra $\mathfrak{Diff}_q$ in a vector space with basis $x^{k-\alpha}$ for $k \in \mathbb{Z}$, where $\mathfrak{Diff}_q$ acts as q-difference operators (see Definition 1.3.1). This representation is called *vector* (or *evaluation*) representation; the parameter u is equal to $q^{-\alpha}$. The Fock module $\mathcal{F}_u$ is isomorphic to $\Lambda^{\infty/2+0}(V_u) \subset \Lambda^{\infty/2}(V_u)$. After the twist, we get a semi-infinite construction of $\mathcal{F}^\sigma_u \subset (\Lambda^{\infty/2} V_u)^\sigma = \Lambda^{\infty/2}(V^\sigma_u)$. Note that conjecturally the semi-infinite construction of $\mathcal{F}^\sigma_u$ can be generalized for $q \neq t$ (cf. [FFJ+11a]).

Twisted W-algebras Denote by $\mathfrak{Diff}^{\geqslant 0}_q$ the subalgebra of $\mathfrak{Diff}_q$ generated by c and $E_{a,b}$, for $a \geqslant 0$. There is an another set of generators $E^k[j]$ of the completion of the $U(\mathfrak{Diff}^{\geqslant 0}_q)$, defined by the formula $\sum_{j\in\mathbb{Z}} E^k[j]z^{-j} = (E(z))^k$ (see Section 1.10 for the definition of the power of $E(z)$). The currents $H(z)$ and $E^k(z)$ for $k \in \mathbb{Z}_{>0}$ satisfy relations of the q-deformed W-algebra of $\mathfrak{gl}_\infty$ (see [Neg18]). We denote this algebra by $\mathcal{W}_q(\mathfrak{gl}_\infty)$.

There is an ideal $J^{\geqslant 0}_{\mu,d}$ in $U(\mathfrak{Diff}^{\geqslant 0}_q) = \mathcal{W}_q(\mathfrak{gl}_\infty)$ which acts by zero on any tensor product $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_d}$, here $\mu = \frac{1}{1-q}(u_1 \cdots u_d)^{1/n}$. This ideal is generated by relations $c = d$ and

$$E^d(z) = \mu^d d! \exp\left(\varphi_-(z)\right) \exp\left(\varphi_+(z)\right), \tag{1.1.9}$$

where

$$\varphi_-(z) = \sum_{j>0} \frac{q^{-j/2} - q^{j/2}}{j} E_{0,-j} z^j, \quad \varphi_+(z) = -\sum_{j>0} \frac{q^{j/2} - q^{-j/2}}{j} E_{0,j} z^{-j}. \tag{1.1.10}$$

The quotient of $\mathcal{W}_q(\mathfrak{gl}_\infty)/J^{\geqslant 0}_{\mu,d}$ is the q-deformed W-algebra of $\mathfrak{gl}_d$. We denote this algebra by $\mathcal{W}_q(\mathfrak{gl}_d)$; it does no depend on μ (up to isomorphism) and acts on any tensor product $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_d}$ (see [FHS+10], [Neg18]).

In Section 1.7 we study a tensor product of the twisted Fock modules $\mathcal{F}^\sigma_{u_1} \otimes \cdots \otimes \mathcal{F}^\sigma_{u_d}$. We prove that the ideal $J^{\geqslant 0}_{\mu,nd,n'd}$ generated by relations $c = nd$ and

$$E^{nd}(z) = z^{n'd} \mu^{nd} (nd)! \exp\left(\varphi_-(z)\right) \exp\left(\varphi_+(z)\right) \tag{1.1.11}$$

acts by zero for $\mu = (-1)^{1/n} \frac{q^{-1/2n}}{q^{1/2}-q^{-1/2}} (u_1 \cdots u_d)^{1/nd}$. We denote the quotient $\mathcal{W}_q(\mathfrak{gl}_\infty)/J^{\geqslant 0}_{\mu,nd,n'd}$ by $\mathcal{W}_q(\mathfrak{gl}_{nd}, n'd)$ and call it the *twisted q-deformed W-algebra of $\mathfrak{gl}_{nd}$*.

There exists another description of the above using the q-deformed W-algebra of $\mathfrak{sl}_n$ introduced in [FF96]. Define $T_k[j]$ by the formula

$$T_k(z) = \sum T_k[j] z^{-j} = \frac{\mu^{-k}}{k!} \exp\left(-\frac{k}{c}\varphi_-(z)\right) E^k(z) \exp\left(-\frac{k}{c}\varphi_+(z)\right). \tag{1.1.12}$$

The generators $T_k[j]$ are elements of a localization of the completion of $U(\mathfrak{Diff}_q^{\geqslant 0})$. These generators commute with H_i and satisfy certain quadratic relations. The algebra generated by $T_k[j]$ is denoted by $\mathcal{W}_q(\mathfrak{sl}_\infty)$.

There is an ideal in $\mathcal{W}_q(\mathfrak{sl}_\infty)$ which acts by zero on any tensor product $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_d}$. This ideal contains relations $c = d$, $T_d(z) = 1$, and $T_{d+k}(z) = 0$ for $k > 0$. The quotient is a standard W-algebra $\mathcal{W}_q(\mathfrak{sl}_d)$ [FF96] (see also Definition 1.7.1). We have a relation $\mathcal{W}_q(\mathfrak{gl}_d) = \mathcal{W}_q(\mathfrak{sl}_d) \otimes U(\mathfrak{Heis})$, where $\mathfrak{Heis}$ is the Heisenberg algebra generated by $E_{0,j}$.

In the case of a product of the twisted Fock modules $\mathcal{F}^\sigma_{u_1} \otimes \cdots \otimes \mathcal{F}^\sigma_{u_d}$ the situation is similar. The corresponding ideal contains the relations $T_{nd}(z) = z^{n'd}$, $T_{nd+k}(z) = 0$ for $k > 0$. We present the quotient in terms of the generators $T_1(z), \ldots, T_{nd}(z)$ and relations (this is Theorem 1.7.1). We call the algebra with such generators and relations by twisted W-algebra $\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)$; see Definition 1.7.2. [1] The quadratic relations in the algebra $\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)$ are the same as in the untwisted case (see eq. (1.7.2)–(1.7.3)), the only difference lies in the relation $T_{nd}(z) = z^{n'd}$.

The algebra $\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)$ is graded, with $\deg T_k[j] = j + \frac{n'k}{n}$. Let us rename the generators by $T^{tw}_k[r] = T_k[r - \frac{n'k}{n}]$, for $r \in \frac{n'k}{n} + \mathbb{Z}$. The presentations of the algebra $\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)$ in terms of generators $T^{tw}_k[r]$ and the presentations of the algebra $\mathcal{W}_q(\mathfrak{sl}_{nd})$ is terms of generators $T_k[r]$ are given by the same formulas; the only difference is the region of r. Heuristically, one can think that $\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)$ is the same algebra as $\mathcal{W}_q(\mathfrak{sl}_{nd})$ but with currents having nontrivial monodromy around zero.

In order to explain these results in more details, consider an example of $\mathfrak{sl}_2$.

Example 1.1.3. As a warm-up, consider the untwisted case $n' = 0$. The algebra $\mathcal{W}_q(\mathfrak{sl}_2)$ is q-deformed Virasoro algebra [SKAO96]. It has one generating current $T(z) = T_1(z)$ and the relation reads

$$\sum_{l=0}^{\infty} f[l]\Big(T[r-l]\,T[s+l] - T[s-l]\,T[r+l]\Big) = -2r(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 \delta_{r+s,0}, \tag{1.1.13}$$

where $f[l]$ are coefficients of a series $\sum_{l=0}^{\infty} f[l] x^l = \sqrt{(1-qx)(1-q^{-1}x)}/(1-x)$. This algebra has a standard bosonization [SKAO96]

$$T(z) = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) z \Big[u :\exp\Big(\eta(q^{1/2}z) - \eta(q^{-1/2}z)\Big): + u^{-1} :\exp\Big(\eta(q^{-1/2}z) - \eta(q^{1/2}z)\Big):\Big], \tag{1.1.14}$$

where $\eta(z) = \sum_{k \neq 0} \eta[k] z^{-k}/k$ and $\eta[k]$ are the generators of the Heisenberg algebra $[\eta[k_1], \eta[k_2]] = \frac{1}{2} k_1 \delta_{k_1+k_2,0}$; one can also add $\eta[0]$ related to the parameter u. In terms of the toroidal algebra $\mathfrak{Diff}_q$ this formula corresponds to the tensor product of two Fock modules $\mathcal{F}_{u_1} \otimes \mathcal{F}_{u_2}$, here $u^2 = u_1/u_2$.

Example 1.1.4. Now, consider the twisted case $n' = 1$. The algebra $\mathcal{W}_q(\mathfrak{sl}_2, 1)$ is generated by one current $T^{tw}(z) = T^{tw}_1(z) = \sum_{r \in \mathbb{Z}+1/2} T^{tw}_1[r] z^{-r}$. The generators $T^{tw}[r] = T^{tw}_1[r]$ satisfy relation (1.1.13). The algebra $\mathcal{W}_q(\mathfrak{sl}_2, 1)$ is called *twisted q-deformed Virasoro algebra*.

As was explained above, the representations of $\mathcal{W}_q(\mathfrak{sl}_2, 1)$ come from the twisted Fock modules $\mathcal{F}^{(1,2)}_u$. The bosonization of the twisted Fock module leads to the bosonization of the $\mathcal{W}_q(\mathfrak{sl}_2, 1)$. Using

[1] One can find a definition of $\mathcal{W}_{q,p}(\mathfrak{sl}_2, 1)$ in [Shi04, (37)–(38)].

formula (1.1.6) we get a bosonization

$$T^{tw}(z) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \left[z^{1/2} :\exp\left(\eta(q^{1/2}z) + \eta(q^{-1/2}z)\right): + z^{3/2} :\exp\left(-\eta(q^{1/2}z) - \eta(q^{-1/2}z)\right):\right]. \tag{1.1.15}$$

Using formula (1.1.7) we get a strange bosonization

$$T^{tw}(z) = (-1)^{\frac{1}{2}} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{2(q^{\frac{1}{4}} - q^{-\frac{1}{4}})} z^{\frac{1}{2}} \left[:\exp\left(\sum_{2\nmid r} \frac{q^{-\frac{r}{4}} - q^{\frac{r}{4}}}{r} J_r z^{-\frac{r}{2}}\right) - :\exp\left(\sum_{2\nmid r} \frac{q^{\frac{r}{4}} - q^{-\frac{r}{4}}}{r} J_r z^{-\frac{r}{2}}\right):\right]. \tag{1.1.16}$$

Here $\eta(z) = \sum_{k\neq 0} \eta[k] z^{-k}/k + Q + \eta[0] \log z$, and J_r are modes of the odd Heisenberg algebra, $[J_r, J_s] = r\delta_{r+s,0}$. These formulas for bosonization are probably new.

Example 1.1.5. One can also use embedding $\mathfrak{Diff}^{\Lambda}_{q^{1/n}} \subset \mathfrak{Diff}_{q^{1/n}}$ in order to construct a bosonization of the W-algebras. Namely one can take a representation of $\mathfrak{Diff}_{q^{1/n}}$ with known bosonization and then express the W-algebra related to $\mathfrak{Diff}_q = \mathfrak{Diff}^{\Lambda}_{q^{1/n}}$ in terms of these bosons.

For example, consider Λ generated by $v_1 = e_1$, $v_2 = 2e_2$ and the Fock representations $\mathcal{F}_{u^{1/2}}$ of $\mathfrak{Diff}_{q^{1/2}}$. One can show (for example, using (1.1.8)) that $\mathcal{W}_q(\mathfrak{gl}_\infty)$ algebra related to $\mathfrak{Diff}_q \cong \mathfrak{Diff}^{\Lambda}_{q^{1/2}}$ acts on $\mathcal{F}_{u^{1/2}}$ through the quotient $\mathcal{W}_q(\mathfrak{gl}_2)$. Therefore, we get an odd bosonization of non-twisted q-deformed Virasoro algebra $\mathcal{W}_q(\mathfrak{sl}_2)$

$$T(z) = \frac{q^{\frac{1}{4}} + q^{-\frac{1}{4}}}{2} \left[:\exp\left(\sum_{2\nmid r} \frac{q^{-\frac{r}{4}} - q^{\frac{r}{4}}}{r} J_r z^{-\frac{r}{2}}\right): + :\exp\left(\sum_{2\nmid r} \frac{q^{\frac{r}{4}} - q^{-\frac{r}{4}}}{r} J_r z^{-\frac{r}{2}}\right):\right]. \tag{1.1.17}$$

Here J_r are the odd modes of the initial boson for $\mathcal{F}_u$. The even modes of the boson disappear in the formula since it belongs to $\mathfrak{Heis} \subset \mathfrak{Diff}^{\Lambda}_{q^{1/2}}$.

It follows from the decomposition (1.1.8) that formula (1.1.17) gives bosonization of certain special representation $\mathcal{W}_q(\mathfrak{sl}_2)$, to be more specific, a direct sum of Fock modules (defined by (1.1.14)) with particular parameters $u = q^{l-1/4}$ for $l \in \mathbb{Z}$.

In the conformal limit $q \to 1$ formula (1.1.17) goes to the odd bosonization of the Virasoro algebra $L_k = \frac{1}{4}\sum_{\frac{1}{2}(r+s)=k} :J_r J_s: + \frac{1}{16}\delta_{k,0}$, see e.g. [Zam87].

Whittaker vectors and relations on conformal blocks As an application, in Section 1.9 we prove the following identity

$$z^{\frac{1}{2}\sum \frac{i^2}{n^2}} \prod_{i\neq j} \frac{1}{(q^{1+\frac{i-j}{n}}; q, q)_\infty} \left(q^{\frac{1}{n}} z^{\frac{1}{n}}; q^{\frac{1}{n}}, q^{\frac{1}{n}}\right)_\infty = \sum_{(l_0,\ldots,l_{n-1})\in \mathbf{Q}_{(n)}} \mathcal{Z}\left(q^{l_0}, q^{\frac{1}{n}+l_1}, \ldots, q^{\frac{n-1}{n}+l_{n-1}}; z\right). \tag{1.1.18}$$

Here the lattice $\mathbf{Q}_{(n)}$ is as above, $(u; q, q)_\infty = \prod_{i,j=0}^{\infty}(1 - q^{i+j}u)$. The function $\mathcal{Z}(u_1, \ldots, u_n; z)$ is a Whittaker limit of conformal block. By AGT relation it equals to the Nekrasov partition function. We recall the definition of $\mathcal{Z}(u_1, \ldots, u_n; z)$ below.

The relation (1.1.18) was conjectured in [BGM19] in the framework of q-isomonodromy/CFT correspondence. As we discussed in the first part of the introduction the main statement of this correspondence is an explicit formula for the q-isomonodromic tau function as an infinite sum of conformal blocks. The left-hand side of (1.1.18) is a tau function corresponding to the algebraic solution of deautonomized discrete flow in Toda system, see [BGM19, eq. (3.11)]. The right-hand side of (1.1.18) is a specialization of conjectural formula [BGM19, eq. (3.6)] for the generic tau function of these flows. In differential case the isomonodromy/CFT correspondanse is proven in many cases, see [ILT15], [BS15], [GL18],[GIL20], but in the q-difference case the main statements are still conjectures.

The generic formula for tau function of deautonomized discrete flow in Toda system is proven only for particular case $n = 2$ [BS19],[MN19]. Here we prove formula for arbitrary n but for special solution.

Let us recall the definition of $\mathcal{Z}(u_1, \ldots, u_n; z)$. The Whittaker vector $W(z|u_1, \ldots, u_N)$ is a vector in a completion of $\mathcal{F}_{u_1} \otimes \ldots \otimes \mathcal{F}_{u_n}$, which is an eigenvector of $E_{a,b}$ for $Nb \geqslant a \geqslant 0$ with certain eigenvalues depending on z, see Definition 1.9.1. Such vector exists and unique for generic values of $u_1, \ldots, u_n$. This property looks to be a part of folklore, we give a proof of this in Section 1.13. The proof is essentially based on the results of [Neg18], [Neg17]. The function $\mathcal{Z}$ is proportional to a Shapovalov pairing of two Whittaker vectors

$$\mathcal{Z}(u_1, \ldots, u_n; z) = z^{\frac{\sum(\log u_i)^2}{2(\log q)^2}} \prod_{i \neq j} \frac{1}{(qu_iu_j^{-1}; q, q)_\infty} \left\langle W_u(1|qu_n^{-1}, \ldots, qu_1^{-1}), W(z|u_1, \ldots, u_n) \right\rangle. \quad (1.1.19)$$

We give a proof of (1.1.18) using decomposition (1.1.8). We consider the Whittaker vector $W(z|1)$ for the algebra $\mathfrak{Diff}_{q^{1/n}}$. Its Shapovalov pairing gives the left-hand side of the relation (1.1.18). On the other hand, we prove that its restriction to summands $\mathcal{F}_{q^{l_0}} \otimes \cdots \otimes \mathcal{F}_{q^{\frac{n-1}{n}+l_{n-1}}}$ is the Whittaker vector for the algebra $\mathfrak{Diff}_q$. So taking the Shapovalov pairing we get the right-hand side of the relation (1.1.18).

In the conformal limit $q \to 1$ the analogue of the relation (1.1.18) in case $n = 2$ was proven in [BS17a] by a similar method. The conformal limit of the decomposition (1.1.8) was studied in [BGM18].

Discussion of $q \neq t$ case. As we mentioned above, $\mathfrak{Diff}_q$ is a specialization of quantum toroidal algebra $U_{q,t}(\ddot{\mathfrak{gl}}_1)$ for $q = t$. It is much more interesting to study the algebra without the constrain. Let us discuss our expectations on generalizations of the results from this chapter.

Fermionic construction (see Theorem 1.4.1) will be generalized after the replacement of the fermions by vertex operators of quantum affine $\mathfrak{gl}_n$ (see Chapter 3). Hence we have bosonization, expressing the currents in terms of exponents dressed by screenings. We also expect that representations of twisted and non-twisted W_n-algebras can be realized via these vertex operators (see Chapter 2 for the $n = 2$ case). It is not clear how one can generalize strange bosonization and connection with isomonodromy/CFT correspondence for $q \neq t$.

Plan of the Chapter. The chapter is organized as follows.

In Section 1.2 we recall basic definitions and properties on the algebra $\mathfrak{Diff}_q$.

In Section 1.3 we recall basic constructions of the Fock module $\mathcal{F}_u$.

In Section 1.4 we present three constructions of the twisted Fock module $\mathcal{F}_u^\sigma$: the fermionic construction in Theorem 1.4.1, the bosonic construction in Theorem 1.4.2, and the strange bosonic construction in Theorem 1.4.3.

In Section 1.5 we study restriction of the Fock module to a subalgebra $\mathfrak{Diff}_q^\Lambda$. Using these restrictions we prove Theorems 1.4.1, 1.4.2, 1.4.3.

In Section 1.6 we give an independent proof of Theorem 1.4.1 using the semi-infinite construction.

In Section 1.7 we study twisted q-deformed W-algebras. We define $\mathcal{W}_q(\mathfrak{sl}_n, n_{tw})$ by generators and relations. Then we show in Theorem 1.7.1 that the tensor product $\mathcal{W}_q(\mathfrak{sl}_n, n_{tw}) \otimes U(\mathfrak{Heis})$ is isomorphic to the certain quotient of $U(\mathfrak{Diff}_q)$; we denote this quotient by $\mathcal{W}_q(\mathfrak{gl}_n, n_{tw})$. We show that $\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)$ acts on the tensor product of twisted Fock modules $\mathcal{F}_{u_1}^\sigma \otimes \cdots \otimes \mathcal{F}_{u_d}^\sigma$. At the end of the section we study relation between these modules and the Verma modules for $\mathcal{W}_q(\mathfrak{gl}_{nd}, n'd)$ and $\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)$.

In Section 1.8 we prove decomposition (1.1.8). Then we study the strange bosonization of W-algebra modules arising from the restriction of Fock module on $\mathfrak{Diff}_q^\Lambda$.

In Section 1.9 we recall definitions and properties of Whittaker vector, Shapovalov pairing, and conformal blocks. Then we prove (1.1.18), see Theorem 1.9.3.

In Section 1.10 we give a definition and study necessary properties of regular product of currents $A(z)B(az)$ for $a \in \mathbb{C}$.

Sections 1.11 and 1.12 consist of calculations which are used in Section 1.7.

In Section 1.13 we study the Whittaker vector for $\mathfrak{Diff}_q$ in the completion of the tensor product $\mathcal{F}_{u_1} \otimes \ldots \otimes \mathcal{F}_{u_n}$. We prove its existence and uniqueness (we use this in Section 1.9). To prove existence we present a construction of Whittaker vector via an intertwiner operator from [AFS12]. We also relate this Whittaker vector to the Whittaker vector of $\mathcal{W}_q(\mathfrak{sl}_n)$ introduced in [Tak10].

1.2 q-difference operators

In this section introduce notation and recall basic facts about algebra $\mathfrak{Diff}_q$, see [FFZ89], [GKL92], and [KR93].

Definition 1.2.1. *The associative algebra of q-difference operators* $\mathrm{Diff}_q^{\mathrm{A}}$ *is an associative algebra generated by* $D^{\pm 1}$ *and* $x^{\pm 1}$ *with the relation* $Dx = qxD$.

Definition 1.2.2. *The algebra of q-difference operators* $\mathfrak{Diff}_q$ *is a Lie algebra with a basis* $E_{k,l}$ *(where* $(k,l) \in \mathbb{Z}^2 \backslash \{(0,0)\}$*),* c *and* c'*. The elements* c *and* c' *are central. All other commutators are given by*

$$[E_{k,l}, E_{r,s}] = (q^{(sk-lr)/2} - q^{(lr-sk)/2})E_{k+r,l+s} + \delta_{k,-r}\,\delta_{l,-s}(c'k + cl). \tag{1.2.1}$$

Remark 1.2.1. Note that the vector subspace of $\mathrm{Diff}_q^{\mathrm{A}}$ spanned by $x^l D^k$ (for $(l,k) \neq (0,0)$) is closed under commutation i.e. has a natural structure of Lie algebra (denote this Lie algebra by $\mathrm{Diff}_q^{\mathrm{L}}$). Consider a basis of this Lie algebra $E_{k,l} := q^{kl/2} x^l D^k$. Finally, $\mathfrak{Diff}_q$ is a central extension of $\mathrm{Diff}_q^{\mathrm{L}}$ by two-dimensional abelian Lie algebra spanned by c and c'.

1.2.1 $SL_2(\mathbb{Z})$ action

In this section we will define action $SL_2(\mathbb{Z})$ on $\mathfrak{Diff}_q$. Let σ be an element of $SL_2(\mathbb{Z})$ corresponding to a matrix

$$\sigma = \begin{pmatrix} m' & m \\ n' & n \end{pmatrix}. \tag{1.2.2}$$

Then σ acts as follows

$$\sigma(E_{k,l}) = E_{m'k+ml,\, n'k+nl}, \quad \sigma(c') = m'c' + n'c, \quad \sigma(c) = mc' + nc. \tag{1.2.3}$$

Proposition 1.2.1. *Formula* (1.2.3) *defines* $SL_2(\mathbb{Z})$ *action on* $\mathfrak{Diff}_q$ *by Lie algebra automorphisms.*

Proof. Note that (1.2.1) is $SL_2(\mathbb{Z})$ covariant. □

For any $\mathfrak{Diff}_q$-module M denote by $\rho_M \colon \mathfrak{Diff}_q \to \mathfrak{gl}(M)$ the corresponding homomorphism.

Definition 1.2.3. *For any* $\mathfrak{Diff}_q$*-module* M *and* $\sigma \in SL(2,\mathbb{Z})$ *let us define the representation* M^σ *as follows.* M *and* M^σ *are the same vector space with different actions, namely* $\rho_{M^\sigma} = \rho_M \circ \sigma$.

We will refer to M^σ as a *twisted representation.* More precisely, M^σ is the representation M, twisted by σ.

1.2.2 Chevalley generators and relations

The Lie algebra $\mathfrak{Diff}_q$ is generated by $E_k := E_{1,k}$, $F_k := E_{-1,k}$ and $H_k := E_{0,k}$. We will call them the *Chevalley generators* of $\mathfrak{Diff}_q$. Define the following *currents* (i.e. formal power series with coefficients in $\mathfrak{Diff}_q$)

$$E(z) = \sum_{k\in\mathbb{Z}} E_{1,k} z^{-k} = \sum_{k\in\mathbb{Z}} E_k z^{-k}; \tag{1.2.4}$$

$$F(z) = \sum_{k\in\mathbb{Z}} E_{-1,k} z^{-k} = \sum_{n\in\mathbb{Z}} F_k z^{-k}; \tag{1.2.5}$$

$$H(z) = \sum_{k\neq 0} E_{0,k} z^{-k} = \sum_{k\neq 0} H_k z^{-k}. \tag{1.2.6}$$

Let us also define the formal delta function

$$\delta(x) = \sum_{k\in\mathbb{Z}} x^k. \tag{1.2.7}$$

Proposition 1.2.2. *Lie algebra $\mathfrak{Diff}_q$ is presented by the generators E_k, F_k (for all $k \in \mathbb{Z}$), H_l (for $l \in \mathbb{Z}\backslash\{0\}$), c, c' and the following relations*

$$[H_k, H_l] = kc\, \delta_{k+l,0}\,; \tag{1.2.8}$$

$$[H_k, E(z)] = (q^{-k/2} - q^{k/2}) z^k E(z), \quad [H_k, F(z)] = (q^{k/2} - q^{-k/2}) z^k F(z); \tag{1.2.9}$$

$$(z - qw)(z - q^{-1}w)[E(z), E(w)] = 0, \quad (z - qw)(z - q^{-1}w)[F(z), F(w)] = 0; \tag{1.2.10}$$

$$[E(z), F(w)] = \left(H(q^{-1/2}w) - H(q^{1/2}w) + c'\right) \delta(w/z) + c\,\frac{w}{z}\delta'(w/z); \tag{1.2.11}$$

$$z_2 z_3^{-1}[E(z_1), [E(z_2), E(z_3)]] + \text{cyclic} = 0; \tag{1.2.12}$$

$$z_2 z_3^{-1}[F(z_1), [F(z_2), F(z_3)]] + \text{cyclic} = 0. \tag{1.2.13}$$

One can find a proof of Proposition 1.2.2 in [Mik07, Thm. 2.1] or [Tsy17, Thm. 5.5].

1.3 Fock module

In this section we review basic constructions of representations of $\mathfrak{Diff}_q$ with $c = 1$ and $c' = 0$. These construction were studied in [GKL92].

1.3.1 Free boson realization

Introduce the Heisenberg algebra generated by a_k (for $k \in \mathbb{Z}$) with relation $[a_k, a_l] = k\delta_{k+l,0}$. Consider the Fock module F^a_α generated by $|\alpha\rangle$ such that $a_k|\alpha\rangle = 0$ for $k > 0$, $a_0|\alpha\rangle = \alpha|\alpha\rangle$.

Proposition 1.3.1. *The following formulas determine an action of $\mathfrak{Diff}_q$ on F^a_α:*

$$c \mapsto 1, \quad c' \mapsto 0, \quad H_k \mapsto a_k; \tag{1.3.1}$$

$$E(z) \mapsto \frac{u}{1-q} \exp\left(\sum_{k>0} \frac{q^{-k/2} - q^{k/2}}{k} a_{-k} z^k\right) \exp\left(\sum_{k<0} \frac{q^{-k/2} - q^{k/2}}{k} a_{-k} z^k\right); \tag{1.3.2}$$

$$F(z) \mapsto \frac{u^{-1}}{1-q^{-1}} \exp\left(\sum_{k>0} \frac{q^{k/2} - q^{-k/2}}{k} a_{-k} z^k\right) \exp\left(\sum_{k<0} \frac{q^{k/2} - q^{-k/2}}{k} a_{-k} z^k\right). \tag{1.3.3}$$

We will denote this representation by $\mathcal{F}_u$.

Remark 1.3.1. Note that α does not appear in formulas (1.3.1)–(1.3.3). But we will need operator a_0 later (see the proof of Proposition 1.3.6) for the boson-fermion correspondence. Heuristically, one can think that $u = q^{-\alpha}$.

Remark 1.3.2 (on our notation). In this chapter, we consider several algebras and their action on the corresponding Fock modules. We choose the following notation. All these representations are denoted by the letter F (for Fock) with some superscript to mention an algebra. Since $\mathfrak{Diff}_q$ is the most important algebra in this chapter, we use no superscript for its representation. Also, let us remark that we consider several copies of the Heisenberg algebra. To distinguish their Fock modules, we write a letter for generators as a superscript.

The standard bilinear form on F^a_α is defined by the following conditions: operator a_{-k} is dual of a_k, the pairing of $|\alpha\rangle$ with itself equals 1. We will use the bra-ket notation for this scalar product. For an operator A we denote by $\langle\alpha|A|\alpha\rangle$ the scalar product of $A|\alpha\rangle$ with $|\alpha\rangle$.

Proposition 1.3.2. *Suppose the algebra $\mathfrak{Diff}_q$ acts on F^a_α so that $H_k \mapsto a_k$ and $\langle\alpha|E(z)|\alpha\rangle = \frac{u}{1-q}$; $\langle\alpha|F(z)|\alpha\rangle = \frac{u^{-1}}{1-q^{-1}}$. Then this representation is isomorphic to $\mathcal{F}_u$.*

Proof. Consider the current

$$T(z) = \exp\left(-\sum_{k>0} \frac{q^{-k/2} - q^{k/2}}{k} a_{-k} z^k\right) E(z)\ \exp\left(-\sum_{k<0} \frac{q^{-k/2} - q^{k/2}}{k} a_{-k} z^k\right).$$

It is easy to verify that $[a_k, T(z)] = 0$. Since F_α is irreducible, $T(z) = f(z)$ for some formal power series $f(z)$ with $\mathbb{C}$-coefficients. On the other hand, $f(z) = \langle\alpha|E(z)|\alpha\rangle = \frac{u}{1-q}$. This implies (1.3.2). The proof of (1.3.3) is analogous. □

Proposition 1.3.3. *Denote $E_l(z) = E_{l,k} z^{-k}$. The action of $E_l(z)$ on Fock representation $\mathcal{F}_u$ is given by the following formula*

$$E_l(z) \to \frac{u^l}{1-q^l} \exp\left(\sum_{k>0} \frac{q^{-kl/2} - q^{kl/2}}{k} a_{-k} z^k\right) \exp\left(\sum_{k<0} \frac{q^{-kl/2} - q^{kl/2}}{k} a_{-k} z^k\right). \tag{1.3.4}$$

Proof. The commutation relation (1.2.1) implies that formula (1.3.4) holds up to a pre-exponential factor. Also, we see from (1.2.1) that

$$E(z)E_l(w) = \frac{q^{-1}w}{z - q^{-1}w} E_{l+1}(q^{-1}w) - \frac{q^l w}{z - q^l w} E_{l+1}(w) + \text{reg}. \tag{1.3.5}$$

The factor can be found inductively from (1.3.5). □

1.3.2 Free fermion realization

In this section we give another construction for the Fock representation of $\mathfrak{Diff}_q$. To do this, let us consider the Clifford algebra, generated by ψ_i and ψ^*_j for $i, j \in \mathbb{Z}$ subject to the relations

$$\{\psi_i, \psi_j\} = 0, \quad \{\psi^*_i, \psi^*_j\} = 0; \tag{1.3.6}$$

$$\{\psi_i, \psi^*_j\} = \delta_{i+j,0}. \tag{1.3.7}$$

Consider the currents

$$\psi(z) = \sum_i \psi_i z^{-i-1}; \quad \psi^*(z) = \sum_i \psi^*_i z^{-i}. \tag{1.3.8}$$

Consider a module F^{ψ} with a cyclic vector $|l\rangle$ and relation

$$\psi_i|l\rangle = 0 \quad \text{for } i \geqslant l, \quad \psi_j^*|l\rangle = 0 \quad \text{for } j > -l. \tag{1.3.9}$$

The module F^{ψ} is independent of l. The isomorphism can be seen from the formulas $\psi_{-l}^*|l\rangle = |l+1\rangle$ and $\psi_{l-1}|l\rangle = |l-1\rangle$. Let us define the l-dependent normal ordered product (to be compatible with $|l\rangle$) by the following formulas

$$:\psi_i\psi_j^*:_{(l)} = -\psi_j^*\psi_i \quad \text{for } i \geqslant l, \tag{1.3.10}$$

$$:\psi_i\psi_j^*:_{(l)} = \psi_i\psi_j^* \quad \text{for } i < l. \tag{1.3.11}$$

Proposition 1.3.4. *The following formulas determine an action of $\mathfrak{Diff}_q$ on F^{ψ}:*

$$c \mapsto 1, \quad c' \mapsto 0, \quad H_k \mapsto \sum_{i+j=k} \psi_i\psi_j^*; \tag{1.3.12}$$

$$E(z) \mapsto \frac{q^l u}{1-q} + uq^{-1/2}z :\psi(q^{-1/2}z)\psi^*(q^{1/2}z):_{(l)} = uq^{-1/2}z\psi(q^{-1/2}z)\psi^*(q^{1/2}z); \tag{1.3.13}$$

$$F(z) \mapsto \frac{q^{-l}u^{-1}}{1-q^{-1}} + u^{-1}q^{1/2}z :\psi(q^{1/2}z)\psi^*(q^{-1/2}z):_{(l)} = u^{-1}q^{1/2}z\psi(q^{1/2}z)\psi^*(q^{-1/2}z). \tag{1.3.14}$$

Let us denote this representation by $\mathcal{M}_u$.

Remark 1.3.3. The Products $\psi(q^{-1/2}z)\psi^*(q^{1/2}z)$ and $\psi(q^{1/2}z)\psi^*(q^{-1/2}z)$ from formulas (1.3.13)–(1.3.14) are not normally ordered (see Section 1.10 for a formal definition and some other technical details on the *regular product*). In particular, this reformulation implies that $\mathcal{M}_u$ does not depend on l.

1.3.3 Semi-Infinite construction

Definition 1.3.1. *The evaluation representation V_u of the algebra $\mathfrak{Diff}_q$ is a vector space with the basis x^k for $k \in \mathbb{Z}$ and the action*

$$E_{a,b}\, x^k = u^a q^{\frac{ab}{2}+ak} x^{k+b}; \qquad c = c' = 0. \tag{1.3.15}$$

Remark 1.3.4. The associative algebra $\mathrm{Diff}_q^{\mathrm{A}}$ acts on V_u. The representation of $\mathfrak{Diff}_q$ is obtained via evaluation homomorphism $\mathrm{ev}\colon \mathfrak{Diff}_q \to \mathrm{Diff}_q^{\mathrm{A}}$.

Remark 1.3.5. Informally, one can consider $x^k \in V_u$ as $x^{k-\alpha}$ for $u = q^{-\alpha}$. Define the action of $\mathrm{Diff}_q^{\mathrm{A}}$ as follows. The generator x acts by multiplication and $Dx^{k-\alpha} = q^{k-\alpha}x^{k-\alpha} = uq^k x^{k-\alpha}$. However, $q^{-\alpha}$ is not well defined for arbitrary complex α. So we consider u as a parameter of representation instead of α.

Let us consider the *semi-infinite exterior power* of the evaluation representation $\Lambda^{\infty/2} V_u$. It is spanned by $|\lambda, l\rangle = x^{l-\lambda_1} \wedge x^{l+1-\lambda_2} \wedge \cdots \wedge x^{l+N} \wedge x^{l+N+1} \wedge x^{l+N+2} \wedge \cdots$ where λ is a Young diagram and $l \in \mathbb{Z}$. Let $p_1 > \ldots > p_i$ and $q_1 > \ldots > q_i$ be Frobenius coordinates of λ.

Proposition 1.3.5. *There is a $\mathfrak{Diff}_q$-modules isomorphism $\Lambda^{\infty/2} V_u \xrightarrow{\sim} \mathcal{M}_u$ given by*

$$|\lambda, l\rangle \mapsto (-1)^{\sum_k (q_k - 1)} \psi_{-p_1+l} \cdots \psi_{-p_i+l}\, \psi^*_{-q_i-l+1} \cdots \psi^*_{-q_1-l+1}|l\rangle. \tag{1.3.16}$$

Proposition 1.3.6. *There is an isomorphism of $\mathfrak{Diff}_q$-modules $\mathcal{M}_u \cong \bigoplus_{l\in\mathbb{Z}} \mathcal{F}_{q^l u}$. The submodule $\mathcal{F}_{q^l u}$ is spanned by $|\lambda, l\rangle$.*

Proof. Recall the ordinary boson-fermion correspondence (see [KR87]). The coefficients of

$$a(z) = \sum_n a_n z^{-n-1} = :\psi(z)\psi^*(z):_{(0)} \tag{1.3.17}$$

are indeed generators of the Heisenberg algebra. Moreover, $F^\psi = \oplus_{l\in\mathbb{Z}} F^a_{-l}$. The highest vector of F^a_{-l} is $|l\rangle$ (in particular, $a_0|l\rangle = -l|l\rangle$). Note that this is the decomposition of $\mathfrak{Diff}_q$-modules as well. Also, note that

$$\langle l|E(z)|l\rangle = \frac{q^l u}{1-q}, \quad \langle l|F(z)|l\rangle = \frac{q^{-l}\,u^{-1}}{1-q^{-1}}.$$

Therefore one can use Proposition 1.3.2 for each summand F^a_{-l}. □

There is a basis in the Fock module $\mathcal{F}_u$ given by semi-infinite monomials

$$|\lambda\rangle = x^{-\lambda_1} \wedge x^{1-\lambda_2} \wedge \cdots \wedge x^{i-\lambda_{i+1}} \wedge \cdots . \tag{1.3.18}$$

To write the action of $\mathfrak{Diff}_q$ in this basis, let us remind the standard notation. Let $l(\lambda)$ be the number of non-zero rows. We will write $s = (i, j)$ for the jth box in the ith row (i.e. $j \leqslant \lambda_i$). The content of a box $c(s) := i - j$. For the diagram $\mu \subset \lambda$, we define a skew Young diagram $\lambda\backslash\mu$, being a set of boxes in λ which are not in μ. Ribbon is a skew Young diagram without 2×2 squares. The height $\mathrm{ht}(\lambda\backslash\mu)$ of a ribbon is one less than the number of its rows.

Proposition 1.3.7. *The action of $\mathfrak{Diff}_q$ on $\mathcal{F}_u$ is given by the following formulas*

$$E_{a,-b}\,|\lambda\rangle = q^{-\frac{a}{2}} u^a \sum_{\mu\backslash\lambda = b-ribbon} (-1)^{\mathrm{ht}(\mu\backslash\lambda)} q^{\frac{a}{b}\sum_{s\in\mu\backslash\lambda} c(s)} |\mu\rangle; \tag{1.3.19}$$

$$E_{a,b}\,|\lambda\rangle = q^{-\frac{a}{2}} u^a \sum_{\lambda\backslash\mu = b-ribbon} (-1)^{\mathrm{ht}(\lambda\backslash\mu)} q^{\frac{a}{b}\sum_{s\in\mu\backslash\lambda} c(s)} |\mu\rangle; \tag{1.3.20}$$

$$E_{a,0}\,|\lambda\rangle = u^a \left(\frac{1}{1-q^a} + \sum_{i=0}^{l(\lambda)-1} \left(q^{a(i-\lambda_{i+1})} - q^{ai} \right) \right) |\lambda\rangle; \tag{1.3.21}$$

here $b > 0$.

In particular,

$$E_{a,0}\ |\varnothing\rangle = \frac{u^a}{1-q^a} |\varnothing\rangle. \tag{1.3.22}$$

Let us introduce the notation $c(\lambda) = \sum_{s\in\lambda} c(s)$. Define an operator $\mathrm{I}_\tau \in \mathrm{End}(\mathcal{F}_u)$ by the following formula

$$\mathrm{I}_\tau|\lambda\rangle = u^{|\lambda|} q^{-\frac{1}{2}|\lambda| + c(\lambda)} |\lambda\rangle \tag{1.3.23}$$

The operator was introduced in [BGHT99] and is well known nowadays.

Proposition 1.3.8. *The operator I_τ enjoys the property $\mathrm{I}_\tau E_{a,b} \mathrm{I}_\tau^{-1} = E_{a-b,b}$.*

Proof. Follows from (1.3.19)–(1.3.21). □

Corollary 1.3.9. *$\mathcal{F}_u^\tau \cong \mathcal{F}_u$ for $\tau = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.*

Remark 1.3.6. Also, Corollary 1.3.9 follows from Proposition 1.3.2: we will use this approach to prove Proposition 1.5.2.

Corollary 1.3.10. *The twisted representation $\mathcal{F}_u^\sigma$ is determined up to isomorphism by n and n'.*

Proof. Corollary 1.3.9 implies that $\mathcal{F}_u^{\tau^k\sigma} \cong \mathcal{F}_u^\sigma$. Note that

$$\tau^k\sigma = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m' & m \\ n' & n \end{pmatrix} = \begin{pmatrix} m' + kn' & m + kn \\ n' & n \end{pmatrix} \tag{1.3.24}$$

For the fixed n and n', all the possible choices of m and m' appear for the appropriate k. □

1.4 Explicit formulas for twisted representation

In this section we provide three explicit constructions of twisted Fock module $\mathcal{F}_u^\sigma$ for

$$\sigma = \begin{pmatrix} m' & m \\ n' & n \end{pmatrix}. \tag{1.4.1}$$

Constructions are called fermionic, bosonic, and strange bosonic. This section contains no proofs. We will give proofs in Sections 1.5. In Section 1.6 we will provide an independent proof of Theorem 1.4.1.

1.4.1 Fermionic construction

We need to consider the $\mathbb{Z}/2\mathbb{Z}$-graded nth tensor power of the Clifford algebra defined above. More precisely, consider an algebra generated by $\psi_{(a)}[i]$ and $\psi^*_{(b)}[j]$, for $i, j \in \mathbb{Z}$; $a, b = 0, \dots, n-1$, subject to relations

$$\left\{\psi_{(a)}[i], \psi_{(b)}[j]\right\} = 0; \quad \left\{\psi^*_{(a)}[i], \psi^*_{(b)}[j]\right\} = 0; \tag{1.4.2}$$

$$\left\{\psi_{(a)}[i], \psi^*_{(b)}[j]\right\} = \delta_{a,b}\ \delta_{i+j,0}. \tag{1.4.3}$$

Consider the currents

$$\psi_{(a)}(z) = \sum_i \psi_{(a)}[i] z^{-i-1}; \quad \psi^*_{(b)}(z) = \sum_i \psi^*_{(b)}[i] z^{-i}. \tag{1.4.4}$$

Consider a module $F^{n\psi}$ with a cyclic vector $|l_0, \dots, l_{n-1}\rangle$ and the relations

$$\psi_{(a)}[i]\ |l_0, \dots, l_{n-1}\rangle = 0 \quad \text{for } i \geqslant l_a, \tag{1.4.5}$$

$$\psi^*_{(a)}[j]\ |l_0, \dots, l_{n-1}\rangle = 0 \quad \text{for } j > -l_a. \tag{1.4.6}$$

The module $F^{n\psi}$ does not depend on $l_0, \dots, l_{n-1}$. The isomorphism can be seen from the following formulas:

$$\psi^*_{(a)}[-l_a]\ |l_0, \dots, l_a, \dots, l_{n-1}\rangle = |l_0, \dots, l_a + 1, \dots, l_{n-1}\rangle, \tag{1.4.7}$$

$$\psi_{(a)}[l_a - 1]\ |l_0, \dots, l_a, \dots, l_{n-1}\rangle = |l_0, \dots, l_a - 1, \dots, l_{n-1}\rangle. \tag{1.4.8}$$

Theorem 1.4.1. *The formulas below determine an action of $\mathfrak{Diff}_q$ on $F^{n\psi}$*

$$c' = n', \quad c = n, \tag{1.4.9}$$

$$H_k^{tw} = \sum_a \sum_{i+j=k} \psi_a[i]\ \psi_a^*[j], \tag{1.4.10}$$

$$E^{tw}(z) = \sum_{b-a \equiv -n' \bmod n} u^{\frac{1}{n}} q^{-1/2} z \psi_{(a)}(q^{-1/2}z) \psi^*_{(b)}(q^{1/2}z) z^{\frac{n'-a+b}{n}} q^{(a+b)/2n}, \tag{1.4.11}$$

$$F^{tw}(z) = \sum_{b-a \equiv n' \bmod n} u^{-\frac{1}{n}} q^{1/2} z \psi_{(a)}(q^{1/2}z) \psi^*_{(b)}(q^{-1/2}z) z^{\frac{-n'-a+b}{n}} q^{-(a+b)/2n}. \tag{1.4.12}$$

The module obtained is isomorphic to $\mathcal{M}_u^\sigma$.

Since $\mathcal{F}_u^\sigma \subset \mathcal{M}_u^\sigma$, we have obtained a fermionic construction for $\mathcal{F}_u^\sigma$.

1.4.2 Bosonic construction

Let us consider the nth tensor power of the Heisenberg algebra. More precisely, this algebra is generated by $a_b[i]$ for $b = 0, \ldots, n-1$ and $i \in \mathbb{Z}$ with the relation $[a_{b_1}[i], a_{b_2}[j]] = i\delta_{b_1,b_2}\delta_{i+j,0}$. Let us extend the algebra by adding the operators e^{Q_b}, obeying the following commutation relations. The operator e^{Q_b} commutes with all the generators except for $a_b[0]$ and satisfy $a_b[0]e^{Q_b} = e^{Q_b}(a_b[0]+1)$. Denote

$$\phi_b(z) = \sum_{j\neq 0} \frac{1}{j} a_b[j] z^{-j} + Q_b + a_b[0]\log z. \tag{1.4.13}$$

Remark 1.4.1. Informally, one can think that there exists an operator Q_b satisfying $[a_b[0], Q_b] = 1$. However, this operator will not act on our representation. We will use Q_b as a formal symbol. Our final answer will consist only of e^{Q_b}, but not of Q_b without the exponent.

We need a notion of a normally ordered exponent $:\exp(\ldots):$. The argument of a normally ordered exponent is a linear combination of $a_b[i]$ and Q_b. Let $\mathtt{a}_+$, $\mathtt{a}_-$, $\mathtt{a}_0$, and $\mathtt{Q}$ denote a linear combination of $a_b[i]$ for $i > 0$, $a_b[i]$ for $i < 0$, $a_b[0]$ and Q_b, correspondingly (b is not fixed). Define

$$:\exp\left(\mathtt{a}_+ + \mathtt{a}_- + \mathtt{a}_0 + \mathtt{Q}\right): \stackrel{\text{def}}{=} \exp\left(\mathtt{a}_+\right)\exp\left(\mathtt{a}_-\right)\exp\left(\mathtt{Q}\right)\exp\left(\mathtt{a}_0\right). \tag{1.4.14}$$

Also, note that $\mathtt{a}_0$ will have the coefficient $\log z$. We shall understand it formally; the action of the operator $\exp(a_b[0]\log z) = z^{a_b[0]}$ is well defined, since in the representation to be considered below, $a_b[0]$ acts as multiplication by an integer at each Fock module.

Let $\mathbf{Q}_{(n)}$ be a lattice with the basis $Q_0 - Q_1, \ldots, Q_{n-2} - Q_{n-1}$. Consider the group algebra $\mathbb{C}[\mathbf{Q}_{(n)}]$. This algebra is spanned by e^λ for $\lambda = \sum_i \lambda_i Q_i \in \mathbf{Q}_{(n)}$. Let us define the action of the commutative algebra generated by $a_b[0]$ on $\mathbb{C}[\mathbf{Q}_{(n)}]$:

$$a_b[0]\, e^{\sum \lambda_i Q_i} = \lambda_b\, e^{\sum \lambda_i Q_i}. \tag{1.4.15}$$

Let F^{na} be the Fock representation of the algebra generated by $a_b[i]$ for $i \neq 0$; i.e. there is a cyclic vector $|\varnothing\rangle \in F^{na}$ such that $a_b[i]\,|\varnothing\rangle = 0$ for $i > 0$.

Finally, we can consider $F^{na} \otimes \mathbb{C}[\mathbf{Q}_{(n)}]$ as representation of the whole Heisenberg algebra as follows: $a_b[i]$ for $i \neq 0$ acts on the first factor, $a_b[0]$ acts on the second factor. Also, $\mathbb{C}[\mathbf{Q}_{(n)}]$ acts on $F^{na} \otimes \mathbb{C}[\mathbf{Q}_{(n)}]$.

Theorem 1.4.2. *There is an action of $\mathfrak{Diff}_q$ on $F^{na} \otimes \mathbb{C}[\mathbf{Q}_{(n)}]$ determined by the formulas*

$$H^{tw}[k] = \sum_b a_b[k], \qquad c' = n', \quad c = n, \tag{1.4.16}$$

$$E^{tw}(z) = \sum_{b-a \equiv -n' \bmod n} u^{\frac{1}{n}} q^{\frac{a+b-n}{2n}} z^{\frac{n'-a+b}{n}+1} :\exp\Big(\phi_b(q^{1/2}z) - \phi_a(q^{-1/2}z)\Big): \epsilon_{a,b}, \tag{1.4.17}$$

$$F^{tw}(z) = \sum_{b-a \equiv n' \bmod n} u^{-\frac{1}{n}} q^{\frac{-a-b+n}{2n}} z^{\frac{-n'-a+b}{n}+1} :\exp\Big(\phi_b(q^{-1/2}z) - \phi_a(q^{1/2}z)\Big): \epsilon_{a,b} \tag{1.4.18}$$

here $\epsilon_{a,b} = \prod_r (-1)^{a_r[0]}$ (we consider the product over such r that $a-1 \geqslant r \geqslant b$ for $a > b$ and $b-1 \geqslant r \geqslant a$ for $b > a$).

The representation obtained is isomorphic to $\mathcal{F}^\sigma_u$.

1.4.3 Strange Bosonic construction

We will use notation of Section 1.3.1. Let ζ be a nth primitive root of unity, e.g. $\zeta = e^{\frac{2\pi i}{n}}$.

Theorem 1.4.3. *There is an action of $\mathfrak{Diff}_q$ on F^a_α determined by the formulas.*

$$H^{tw}_k = a_{nk}, \quad c = n, \quad c' = n', \tag{1.4.19}$$

$$E^{tw}(z) = z^{n'/n} \frac{u^{\frac{1}{n}}}{n(1-q^{1/n})} \sum_{l=0}^{n-1} \zeta^{ln'} :\exp\left(\sum_k \frac{q^{-k/2n} - q^{k/2n}}{k} a_k \zeta^{-kl} z^{-k/n}\right):, \tag{1.4.20}$$

$$F^{tw}(z) = z^{-n'/n} \frac{u^{-\frac{1}{n}}}{n(1-q^{-1/n})} \sum_{l=0}^{n-1} \zeta^{-ln'} :\exp\left(\sum_k \frac{q^{k/2n} - q^{-k/2n}}{k} a_k \zeta^{-kl} z^{-k/n}\right):. \tag{1.4.21}$$

The representation obtained is isomorphic to $\mathcal{F}^\sigma_u$.

As before the representation does not depend on α, see Remark 1.3.1.

1.5 Twisted representation via a sublattice

1.5.1 Sublattices and subalgebras

Consider a full rank sublattice $\Lambda \subset \mathbb{Z}^2$ of index n (i.e. $\mathbb{Z}^2/\Lambda$ is a finite group of order n). Let us define a Lie subalgebra $\mathfrak{Diff}^\Lambda_q \subset \mathfrak{Diff}_q$ which is spanned by $E_{a,b}$ for $(a,b) \in \Lambda$ and central elements c, c'.

Denote by $E^{[n]}_{a,b}$, $c_{[n]}, c'_{[n]}$ standard generators of $\mathfrak{Diff}_{q^n}$. Let $v_1 = (k_1, l_1)$ and $v_2 = (k_2, l_2)$ be a basis of Λ. Define a map $\varphi_{v_1,v_2} \colon \mathfrak{Diff}_{q^n} \to \mathfrak{Diff}_q$

$$\varphi_{v_1,v_2} E^{[n]}_{a,b} = E_{ak_1+bk_2,\, al_1+bl_2}, \tag{1.5.1}$$

$$\varphi_{v_1,v_2} c'_{[n]} = k_1 c' + l_1 c, \tag{1.5.2}$$

$$\varphi_{v_1,v_2} c_{[n]} = k_2 c' + l_2 c. \tag{1.5.3}$$

Proposition 1.5.1. *Let v_1, v_2 be a positively oriented basis (i.e. $k_1 l_2 - k_2 l_1 = n$). Then the map φ_{v_1,v_2} is a Lie algebra isomorphism $\mathfrak{Diff}_{q^n} \cong \mathfrak{Diff}^\Lambda_q$.*

Proof. It follows from (1.2.1) directly. □

Slightly abusing notation, denote the Fock representation of $\mathfrak{Diff}_{q^n}$ by $\mathcal{F}^{[n]}_u$.

Proposition 1.5.2. *Let $v_1 = (n, 0)$ and $v_2 = (-m, 1)$. Then $\mathcal{F}^{[n]}_{u^n} \cong \mathcal{F}_u|_{\phi_{v_1,v_2}(\mathfrak{Diff}_{q^n})}$ as $\mathfrak{Diff}_{q^n}$-modules.*

Proof. Note that the Fock module $\mathcal{F}_u$ is $\mathbb{Z}$-graded with grading given by

$$\deg(|\alpha\rangle) = 0 \quad \deg(E_{a,b}) = -b \tag{1.5.4}$$

Recall that a character of a $\mathbb{Z}$-graded module is the generating function of dimensions of the graded components. Then the character of Fock module $\operatorname{ch} \mathcal{F}_u = 1/(\mathfrak{q})_\infty := \prod_{k=1}^\infty 1/(1-\mathfrak{q}^k)$.

Consider a subalgebra $\mathfrak{Heis}_0$ in $\mathfrak{Diff}_{q^n}$ spanned by $E_{0,k}$ and c. Note that $\mathfrak{Heis}_0$ is isomorphic to the Heisenberg algebra. Since $\deg(\phi_{v_1,v_2} E_{0,-k}) = \deg(E_{km,-k}) = k$, the character of the $\mathfrak{Heis}_0$-Fock module is also $1/(\mathfrak{q})_\infty$; i.e. it coincides with $\operatorname{ch} \mathcal{F}_u$. This implies that $\mathcal{F}_u|_{\phi_{v_1,v_2}(\mathfrak{Diff}_{q^n})}$ restricted to $\mathfrak{Heis}_0$ is isomorphic to the $\mathfrak{Heis}_0$-Fock module. To finish the proof, we use Propositions 1.3.2 and 1.3.3. □

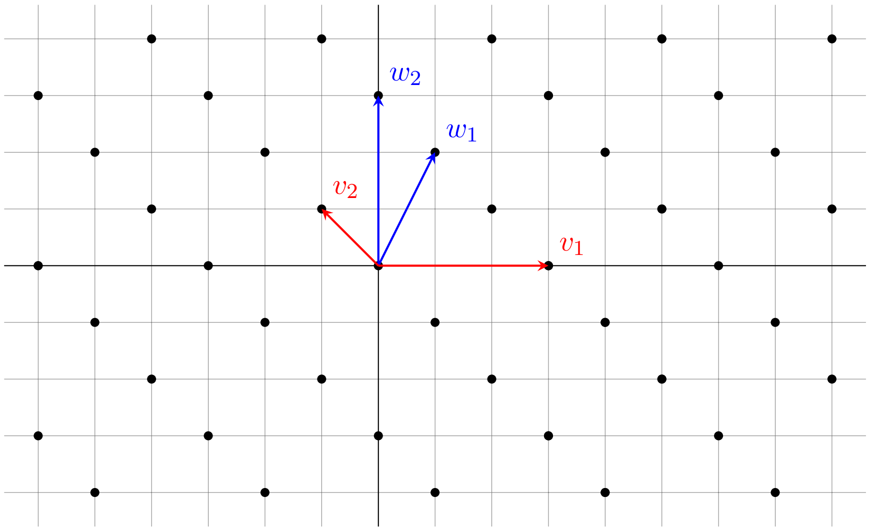


Figure 1.1: Lattice Λ_σ for $n = 3, m = 1$,

1.5.2 Twisted Fock vs restricted Fock

From now on we change $q \to q^{1/n}$. Our goal is to construct an action of $\mathfrak{Diff}_q$ on the Fock module twisted by $\sigma \in SL_2(\mathbb{Z})$ as in (1.4.1) for $n \neq 0$. Consider a sublattice $\Lambda_\sigma \subset \mathbb{Z}^2$ spanned by $v_1 = (n, 0)$ and $v_2 = (-m, 1)$. Consider another basis of Λ_σ obtained by σ

$$w_1 = m'v_1 + n'v_2 = (m'n - n'm, n') = (1, n') \tag{1.5.5}$$
$$w_2 = mv_1 + nv_2 = (0, n) \tag{1.5.6}$$

Remark 1.5.1. The construction of the sublattice $\Lambda_\sigma \subset \mathbb{Z}^2$ naturally appears, if one require σ to be a transition matrix from v_i to w_i and assume $v_1 = (n, 0)$, $w_2 = (0, n)$.

Denote the Fock module of $\mathfrak{Diff}_{q^{1/n}}$ by $\mathcal{F}_u^{[1/n]}$.

Theorem 1.5.1. *There is an isomorphism of $\mathfrak{Diff}_q$-modules $(\mathcal{F}_u)^\sigma \cong \mathcal{F}_{u^{1/n}}^{[1/n]}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)}$*

Proof. Proposition 1.5.2 implies $\mathcal{F}_{u^{1/n}}^{[1/n]}\Big|_{\phi_{v_1,v_2}(\mathfrak{Diff}_q)} \cong \mathcal{F}_u$. On the other hand, relations (1.5.5) and (1.5.6) yield that σ is the transition matrix from w_1, w_2 to v_1, v_2. □

Corollary 1.5.3. *There is an isomorphism of $\mathfrak{Diff}_q$-modules $(\mathcal{M}_u)^\sigma \cong \mathcal{M}_{u^{1/n}}^{[1/n]}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)}$*

Theorem 1.5.1 combined with results from Section 1.3 enables us to find explicit formulas for action on $\mathcal{F}_u^\sigma$. We will do this below.

1.5.3 Explicit formulas for restricted Fock

Fermionic construction via sublattice

Denote fermionic representation of $\mathfrak{Diff}_{q^{1/n}}$ by $\mathcal{M}_u^{[1/n]}$. To be more specific, let us rewrite formulas from Section 1.3.2 for $\mathcal{M}_{u^{1/n}}^{[1/n]}$.

$$c \mapsto 1, \quad c' \mapsto 0, \quad H_m \to \sum_{i+j=m} \psi_i \psi_j^*, \tag{1.5.7}$$

$$E(z) \mapsto \frac{q^{l/n} u^{1/n}}{1-q^{1/n}} + u^{1/n} q^{-1/2n} z :\psi(q^{-1/2n}z)\psi^*(q^{1/2n}z):_{(l)} \tag{1.5.8}$$

$$F(z) \mapsto \frac{q^{-l/n} u^{-1/n}}{1-q^{-1/n}} + u^{-1/n} q^{1/2n} z :\psi(q^{1/2n}z)\psi^*(q^{-1/2n}z):_{(l)} \tag{1.5.9}$$

Proposition 1.5.4. *The following formulas below determine an action of $\mathfrak{Diff}_q$ on $F^{n\psi}$*

$$c = n, \quad c' = n_{tw}, \quad H_k^{tw} = \sum_a \sum_{i+j=k} \psi_a[i]\ \psi_a^*[j], \tag{1.5.10}$$

$$E^{tw}(z) = \sum_{b-a \equiv -n_{tw} \bmod n} u^{\frac{1}{n}} q^{-1/2} z \psi_{(a)}(q^{-1/2}z) \psi_{(b)}^*(q^{1/2}z) z^{\frac{n_{tw}-a+b}{n}} q^{(a+b)/2n}, \tag{1.5.11}$$

$$F^{tw}(z) = \sum_{b-a \equiv n_{tw} \bmod n} u^{-\frac{1}{n}} q^{1/2} z \psi_{(a)}(q^{1/2}z) \psi_{(b)}^*(q^{-1/2}z) z^{\frac{-n_{tw}-a+b}{n}} q^{-(a+b)/2n}. \tag{1.5.12}$$

The module obtained is isomorphic to $\mathcal{M}_{u^{1/n}}^{[1/n]}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)}$ *for* $w_1 = (1, n_{tw})$, $w_2 = (0, n)$.

Remark 1.5.2. Below we will substitute $n_{tw} = n'$ to prove Theorem 1.4.1. However, Proposition 1.5.4 is more general, than it is necessary for the proof, since we do not assume here that $\gcd(n, n_{tw}) = 1$. We will need the case of arbitrary n_{tw} in Section 1.8.

Proof. We use the notation $E(z), F(z)$ and $H(z)$ for the Chevalley generators of $\mathfrak{Diff}_{q^{1/n}}$. The generators of $\mathfrak{Diff}_q \cong \mathfrak{Diff}_{q^{1/n}}^{\Lambda}$ (identified by φ_{w_1,w_2}) will be denoted by $E^{tw}(z), F^{tw}(z)$ and $H^{tw}(z)$. Let us write the identification ϕ_{w_1,w_2} explicitly for the Chevalley generators

$$H^{tw}(z) = \sum_k H_{0,nk} z^{-k} \tag{1.5.13}$$

$$E^{tw}(z) = z^{n_{tw}/n} \sum_{k \equiv n_{tw} \bmod n} E_k z^{-k/n} \tag{1.5.14}$$

$$F^{tw}(z) = z^{-n_{tw}/n} \sum_{k \equiv -n_{tw} \bmod n} F_k z^{-k/n} \tag{1.5.15}$$

Let us consider currents $\psi_{(a)}(z)$ and $\psi_{(b)}^*(z)$ for $a, b = 0, 1, \ldots, n-1$. These currents are defined by following equality

$$z\psi(z) = \sum_{a=0}^{n-1} z^{n-a} \psi_{(a)}(z^n) \qquad \psi^*(z) = \sum_{b=0}^{n-1} z^b \psi_{(b)}^*(z^n) \tag{1.5.16}$$

Let us denote the modes of $\psi_{(a)}(z)$ and $\psi_{(b)}^*(z)$ as in equality (1.4.4). It is easy to see that these modes satisfy Clifford algebra relations (1.4.2), (1.4.3). So we have identified the Clifford algebra and the nth power of Clifford algebra. This leads to an identification $F^{\psi} = F^{n\psi}$.

Substituting (1.5.16) into (1.5.8) and (1.5.9), we obtain

$$E(z) = \frac{q^{l/n}u^{1/n}}{1-q^{1/n}} + \sum_{a=0}^{n-1}\sum_{b=0}^{n-1} u^{1/n} q^{\frac{a+b}{2n}} q^{-1/2} z^{n-a+b} :\psi_{(a)}(q^{-1/2}z^n)\psi^*_{(b)}(q^{1/2}z^n):_{(l)}, \tag{1.5.17}$$

$$F(z) = \frac{q^{-l/n}u^{-1/n}}{1-q^{-1/n}} + \sum_{a=0}^{n-1}\sum_{b=0}^{n-1} u^{-1/n} q^{-\frac{a+b}{2n}} q^{1/2} z^{n-a+b} :\psi_{(a)}(q^{1/2}z^n)\psi^*_{(b)}(q^{-1/2}z^n):_{(l)} . \tag{1.5.18}$$

For technical reasons, we need to treat the cases $n_{tw} \neq 0$ and $n_{tw} = 0$ separately. Let us first consider the case $n_{tw} \neq 0$. Using formulas (1.5.13)–(1.5.15), we see that

$$E^{tw}(z) = \sum_{a-b\equiv n_{tw} \bmod n} q^{\frac{a+b}{2n}} u^{1/n} z^{\frac{n_{tw}-a+b}{n}} q^{-1/2} z \psi_{(a)}(q^{-1/2}z)\psi^*_{(b)}(q^{1/2}z), \tag{1.5.19}$$

$$F^{tw}(z) = \sum_{a-b\equiv -n_{tw} \bmod n} q^{\frac{-(a+b)}{2n}} u^{-1/n} z^{\frac{-n_{tw}-a+b}{n}} q^{1/2} z \psi_{(a)}(q^{1/2}z)\psi^*_{(b)}(q^{-1/2}z). \tag{1.5.20}$$

For $n_{tw} = 0$ we obtain

$$E^{tw}(z) = \frac{q^{l/n}u^{1/n}}{1-q^{1/n}} + \sum_{a=0}^{n-1} u^{1/n} q^{a/n} q^{-1/2} z :\psi_{(a)}(q^{-1/2}z)\psi^*_{(a)}(q^{1/2}z):_{(l)} \tag{1.5.21}$$

$$F^{tw}(z) = \frac{q^{-l/n}u^{-1/n}}{1-q^{-1/n}} + \sum_{a=0}^{n-1} u^{-1/n} q^{-a/n} q^{1/2} z :\psi_{(a)}(q^{1/2}z)\psi^*_{(a)}(q^{-1/2}z):_{(l)} \tag{1.5.22}$$

This can be rewritten as

$$E^{tw}(z) = \sum_{a=0}^{n-1} u^{1/n} q^{a/n} \left(\frac{q^{\lceil \frac{l-a}{n} \rceil}}{1-q} + q^{-1/2} z :\psi_{(a)}(q^{-1/2}z)\psi^*_{(a)}(q^{1/2}z):_{(l)} \right), \tag{1.5.23}$$

$$F^{tw}(z) = \sum_{a=0}^{n-1} u^{-1/n} q^{-a/n} \left(\frac{q^{-\lceil \frac{l-a}{n} \rceil}}{1-q^{-1}} + q^{1/2} z :\psi_{(a)}(q^{1/2}z)\psi^*_{(a)}(q^{-1/2}z):_{(l)} \right). \tag{1.5.24}$$

Note that the l-dependent normal ordering is defined in terms of ψ_i and ψ^*_j. One can check (cf. (1.10.6))

$$\psi_{(a)}(z)\,\psi^*_{(a)}(w) = \frac{(w/z)^{\lceil \frac{l-a}{n} \rceil}}{z(1-w/z)} + :\psi_{(a)}(z)\psi^*_{(a)}(w):_{(l)} . \tag{1.5.25}$$

Hence

$$q^{-1/2} z \psi_{(a)}(q^{-1/2}z)\,\psi^*_{(a)}(q^{1/2}z) = \frac{q^{\lceil \frac{l-a}{n} \rceil}}{1-q} + q^{-1/2} z :\psi_{(a)}(q^{-1/2}z)\psi^*_{(a)}(q^{1/2}z):_{(l)} . \tag{1.5.26}$$

□

Proof of Theorem 1.4.1. Follows from Theorem 1.5.1 and Proposition 1.5.4. □

Bosonic construction via sublattices

Proposition 1.5.5. *There is an action of $\mathfrak{Diff}_q$ on $F^{na} \otimes \mathbb{C}[\mathbf{Q}_{(n)}]$ determined by the following formulas*

$$H^{tw}[k] = \sum_b a_b[k], \quad c' = n_{tw}, \quad c = n, \tag{1.5.27}$$

$$E^{tw}(z) = \sum_{b-a\equiv -n_{tw} \bmod n} u^{\frac{1}{n}} q^{\frac{a+b-n}{2n}} z^{\frac{n_{tw}-a+b}{n}+1} :\exp\Big(\phi_b(q^{1/2}z) - \phi_a(q^{-1/2}z)\Big): \epsilon_{a,b}, \tag{1.5.28}$$

$$F^{tw}(z) = \sum_{b-a\equiv n_{tw} \bmod n} u^{-\frac{1}{n}} q^{\frac{-a-b+n}{2n}} z^{\frac{-n_{tw}-a+b}{n}+1} :\exp\Big(\phi_b(q^{-1/2}z) - \phi_a(q^{1/2}z)\Big): \epsilon_{a,b}. \tag{1.5.29}$$

here $\epsilon_{a,b} = \prod_r (-1)^{a_r[0]}$ *(we consider the product over such* r *that* $a-1 \geqslant r \geqslant b$ *for* $a > b$ *and* $b-1 \geqslant r \geqslant a$ *for* $b > a$*).*

The representation obtained is isomorphic to $\mathcal{F}^{[1/n]}_{u^{1/n}}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)}$ *for* $w_1 = (1, n_{tw})$*,* $w_2 = (0, n)$*.*

Proof Proposition 1.5.5. We need an upgraded version boson-fermion correspondence for the proof. Namely, there is an action of nth tensor power of the Heisenberg algebra on $F^{n\psi}$ given by

$$\partial\phi_b(z) =: \psi_{(b)}(z)\psi^*_{(b)}(z):_{(0)} \tag{1.5.30}$$

Let $\mathbf{P}_{(n)}$ be a lattice spanned Q_b. According to boson-fermion correspondence $F^{n\psi} \cong \bigoplus F^{na} \otimes \mathbb{C}[\mathbf{P}_{(n)}]$.

Lemma 1.5.6. *Vector subspace* $F^{na} \otimes \mathbb{C}[\mathbf{Q}_{(n)}] \subset F^{na} \otimes \mathbb{C}[\mathbf{P}_{(n)}] = F^{n\psi}$ *is a* $\mathfrak{Diff}_q$*-submodule (with respect to action, defined in Proposition 1.5.4). The action of* $\mathfrak{Diff}_q$ *on the subrepresentation is given by* (1.5.27)–(1.5.29)*.*

Proof. One should substitute

$$\psi_{(b)}(z) = :\exp\left(-\phi_b(z)\right): \ (-1)^{\sum_{k=0}^{b-1} a_k[0]} \tag{1.5.31}$$

$$\psi^*_{(b)}(z) = :\exp\left(\phi_b(z)\right): \ (-1)^{\sum_{k=0}^{b-1} a_k[0]} \tag{1.5.32}$$

into fermionic formulas (1.5.10)–(1.5.12). □

Recall that decomposition of $\mathcal{M}^{[1/n]}_u$ is given by eigenvalues of $a[0]$; more precisely, operator $a[0]$ acts by $-j$ on $\mathcal{F}^{[1/n]}_{q^{j/n}u}$.

Lemma 1.5.7. *Using identification* $F^{\psi} = F^{n\psi}$ (cf. (1.5.16)), we obtain $a[0] = a_0[0] + \cdots + a_{n-1}[0]$.

Sketch of the proof. This follows from $\lceil \frac{l-b}{n} \rceil = 0$ for $l = 0$ and $b = 0, \ldots n-1$ (cf. (1.5.25)). □

Lemma 1.5.7 implies that the identification of vector spaces $F^{\psi} = F^{n\psi}$ leads to identification of subspaces $F^a_0 = F^{na} \otimes \mathbb{C}[\mathbf{Q}_{(n)}]$. Let us package identifications of vector subspaces into a commutative diagram

$$\begin{tikzcd}
\mathcal{F}^{[1/n]}_{u^{1/n}}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)} \arrow[r, equal] \arrow[d, hook] & \mathcal{F}^{[1/n]}_{u} \arrow[r, equal] \arrow[d, hook] & F^a_0 \arrow[r, equal] \arrow[d, hook] & F^{na} \otimes \mathbb{C}[\mathbf{Q}_{(n)}] \arrow[d, hook] \\
\mathcal{M}^{[1/n]}_{u^{1/n}}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)} \arrow[r, equal] & \mathcal{M}^{[1/n]}_{u} \arrow[r, equal] & F^{\psi} \arrow[r, equal] & F^{n\psi}
\end{tikzcd}$$

Proposition 1.5.4 states that formulas (1.5.10)–(1.5.12) gives an action of $\mathfrak{Diff}_q$ with respect to identification of bottom line of the diagram. Therefore, Lemma 1.5.6 implies that formulas (1.5.27)–(1.5.29) describes the action of $\mathfrak{Diff}_q$ with respect to identification of top line of the diagram. □

Proof of Theorem 1.4.2. Follows from Theorem 1.5.1 and Proposition 1.5.5. □

Strange bosonic construction via sublattices

Proposition 1.5.8. *There is an action of $\mathfrak{Diff}_q$ on F^a_α defined by formulas.*

$$c = n, \quad c' = n_{tw}, \quad H^{tw}_k = a_{nk}, \tag{1.5.33}$$

$$E^{tw}(z) = z^{n_{tw}/n} \frac{u^{\frac{1}{n}}}{n(1-q^{1/n})} \sum_{l=0}^{n-1} \zeta^{ln_{tw}} :\exp\left(\sum_k \frac{q^{-k/2n} - q^{k/2n}}{k} a_k \zeta^{-kl} z^{-k/n} \right):, \tag{1.5.34}$$

$$F^{tw}(z) = z^{-n_{tw}/n} \frac{u^{-\frac{1}{n}}}{n(1-q^{-1/n})} \sum_{l=0}^{n-1} \zeta^{-ln_{tw}} :\exp\left(\sum_k \frac{q^{k/2n} - q^{-k/2n}}{k} a_k \zeta^{-kl} z^{-k/n} \right): . \tag{1.5.35}$$

Obtained module is isomorphic to $\mathcal{F}^{[1/n]}_{u^{1/n}}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)}$ for $w_1 = (1, n_{tw})$, $w_2 = (0, n)$.

Proof. Formulas (1.5.13)–(1.5.15) imply

$$H^{tw}_k = H_{nk}, \tag{1.5.36}$$

$$E^{tw}(z) = \frac{1}{n} z^{n_{tw}/n} \sum_{l=0}^{n-1} \zeta^{ln_{tw}} E(\zeta^l z^{1/n}), \tag{1.5.37}$$

$$F^{tw}(z) = \frac{1}{n} z^{-n_{tw}/n} \sum_{l=0}^{n-1} \zeta^{-ln_{tw}} F(\zeta^l z^{1/n}). \tag{1.5.38}$$

Substitution of $\mathfrak{Diff}_{q^{1/n}}$-version of (1.3.1)–(1.3.3) to (1.5.36)–(1.5.38) finishes the proof. □

Proof of Theorem 1.4.3. Follows from Theorem 1.5.1 and Proposition 1.5.8. □

1.6 Twisted representation via a Semi-infinite construction

This section is devoted to another proof of the Theorem 1.4.1. So we use the same notation

$$\sigma = \begin{pmatrix} m' & m \\ n' & n \end{pmatrix}$$

Twisted evaluation representation Let $e_{a,b}$ be a matrix unit (all entries are 0 except for one cell, where it is 1; this cell is in bth column and ath row).

Consider a homomorphism $\mathfrak{t}_{u,\sigma} : \mathrm{Diff}^{\mathrm{A}}_q \to \mathrm{Diff}^{\mathrm{A}}_q \otimes \mathrm{Mat}_{n\times n}$ defined by

$$E_{0,k} \mapsto E_{0,k} \otimes 1 \tag{1.6.1}$$

$$E_{1,k} \mapsto u^{\frac{1}{n}} \sum_{b-a \equiv -n' \mod n} q^{\frac{a+b}{2n}} E_{1,k+\frac{b-a+n'}{n}} \otimes e_{a,b} \tag{1.6.2}$$

$$E_{-1,k} \mapsto u^{-\frac{1}{n}} \sum_{b-a \equiv n' \mod n} q^{-\frac{a+b}{2n}} E_{-1,k+\frac{b-a-n'}{n}} \otimes e_{a,b}. \tag{1.6.3}$$

Algebra $\mathrm{Diff}^{\mathrm{A}}_q \otimes \mathrm{Mat}_{n\times n}$ tautologically acts on $\mathbb{C}^n[z, z^{-1}]$. Therefore, homomorphism $\mathfrak{t}_{u,\sigma}$ induces an action of $\mathrm{Diff}^{\mathrm{A}}_q$ on $\mathbb{C}^n[z, z^{-1}]$.

Proposition 1.6.1. *Obtained representation of $\mathrm{Diff}^{\mathrm{A}}_q$ is isomorphic to V^σ_u.*

Proof. Consider a basis $v_l := q^{\frac{ml^2}{2n}} u^{\frac{ml}{n}} x^l$ of evaluation representation $\mathbb{C}[x, x^{-1}]^\sigma$. Action with respect to this basis looks like

$$E_{0,k} v_l = v_{k+l} \tag{1.6.4}$$

$$E_{1,k} v_l = u^{\frac{1}{n}} q^{\frac{n'+kn+2l}{2n}} v_{l+nk+n'} \tag{1.6.5}$$

$$E_{-1,k} v_l = u^{-\frac{1}{n}} q^{\frac{n'-kn-2l}{2n}} v_{l+nk-n'} \tag{1.6.6}$$

Let $a, b = 0, \ldots, n-1$ be such numbers that $l = nj + b$ and $a \equiv b + n' \mod n$. Substituting $l = nj + b$ into (1.6.5) we obtain

$$E_{1,k} v_{nj+b} = u^{\frac{1}{n}} q^{\frac{a+b}{2n}} q^{\frac{n'+b-a}{2n}} q^{\frac{k}{2}} q^j v_{n(k+j+\frac{n'+b-a}{n})+a} \tag{1.6.7}$$

Let us identify $\mathbb{C}^n[z, z^{-1}] \xrightarrow{\sim} \mathbb{C}[x, x^{-1}]$ by $z^j e_b \mapsto v_{nj+b}$. Then formula (1.6.7) will be rewritten

$$E_{1,k}(z^j e_b) = u^{\frac{1}{n}} q^{\frac{a+b}{2n}} \left(E_{1,k+\frac{b-a+n'}{n}} \otimes e_{a,b} \right) (z^j e_b) \tag{1.6.8}$$

To be compared with formula (1.6.2) this proves the proposition for $E_{1,k}$. The proof for $E_{-1,k}$ is analogous. For $E_{0,k}$ proposition is obvious from (1.6.4). □

Semi-infinite construction. To apply semi-infinite construction we need to pass from associative algebras to Lie algebras.

Definition 1.6.1. *Algebra $\mathfrak{Diff}_q(\mathfrak{gl}_n)$ is a Lie algebra with basis $E_{k,l} \otimes e_{a,b}$ (where $(k,l) \in \mathbb{Z}^2 \backslash (0,0)$ and $a, b = 0, \ldots, n-1$), c and c'. Elements c and c' are central. All other commutators are given by*

$$[E_{k_1,l_1} \otimes e_{a_1,b_1}, E_{k_2,l_2} \otimes e_{a_2,b_2}] = E_{k_1+k_2,l_1+l_2} \otimes \left(q^{\frac{l_2k_1-l_1k_2}{2}} \delta_{b_1,a_2} e_{a_1,b_2} - q^{\frac{l_1k_2-l_2k_1}{2}} \delta_{b_2,a_1} e_{a_2,b_1} \right) + \\ + \delta_{k_1,-k_2} \delta_{l_1,-l_2} \delta_{a_2,b_1} \delta_{a_1,b_2} (cl_1 + c'k_1). \tag{1.6.9}$$

Proposition 1.6.2. *There is an action of $\mathfrak{Diff}_q(\mathfrak{gl}_n)$ on $F^{n\psi}$ given by formulas*

$$c \mapsto 1; \quad c' \mapsto 0 \tag{1.6.10}$$

$$E(z) \otimes e_{a,b} \mapsto q^{-\frac{1}{2}} z \psi_{(a)}(q^{-\frac{1}{2}} z) \psi^*_{(b)}(q^{\frac{1}{2}} z) \tag{1.6.11}$$

$$F(z) \otimes e_{a,b} \mapsto q^{\frac{1}{2}} z \psi_{(a)}(q^{\frac{1}{2}} z) \psi^*_{(b)}(q^{-\frac{1}{2}} z) \tag{1.6.12}$$

Obtained representation is isomorphic to $\Lambda^{\infty/2} \mathbb{C}^n[z, z^{-1}]$.

Proof of Theorem 1.4.1. According to Proposition 1.3.5, $\mathcal{M}_u \cong \Lambda^{\infty/2} V_u$. Then $\mathcal{M}_u^\sigma \cong \Lambda^{\infty/2} V_u^\sigma$. Therefore, Propositions 1.6.1 and 1.6.2 imply Theorem 1.4.1 □

1.7 q-W-Algebras

1.7.1 Definitions

Topological algebras and completions In this section we will work with topological algebras. Let us define topological algebra appearing as a completion of $\mathfrak{Diff}_q$. It is given by projective limit of $U(\mathfrak{Diff}_q)/J_k$ where J_k is the left ideal generated by non-commutative polynomials in E_{j_1,j_2} of degree $-k$ (with respect to grading $\deg E_{j_1,j_2} = -j_2$). Although each $U(\mathfrak{Diff}_q)/J_k$ does not have a structure of algebra, so does the projective limit. Moreover, the projective limit has natural topology.

Below we will ignore all corresponding technical problems concerning completions and topology. We will use term 'generators' instead of 'topological generators', the same notation for $\mathfrak{Diff}_q$, and its completion and so on.

Non-Twisted W-Algebras

Let us introduce a notation

$$\sum_{l=0}^{\infty} f_{k,n}[l]x^l = f_{k,n}(x) = \frac{(1-qx)^{\frac{n-k}{n}}(1-q^{-1}x)^{\frac{n-k}{n}}}{(1-x)^{\frac{2(n-k)}{n}}} \tag{1.7.1}$$

Definition 1.7.1. *Algebra $\mathcal{W}_q(\mathfrak{sl}_n)$ is generated by $T_k[r]$ for $r \in \mathbb{Z}$ and $k = 1, \ldots, n-1$. It is convenient to add generators $T_0[r] = T_n[r] = \delta_{r,0}$. The defining relations are*

$$\sum_{l=0}^{\infty} f_{k,n}[l]\Big(T_1[r-l]T_k[s+l] - T_k[s-l]T_1[r+l]\Big) = -(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2(kr-s)T_{k+1}[r+s] \tag{1.7.2}$$

$$\sum_{l=0}^{\infty} f_{n-k,n}[l]\Big(T_{n-1}[r-l]T_k[s+l] - T_k[s-l]T_{n-1}[r+l]\Big) = -(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2\big((n-k)r-s\big)T_{k-1}[r+s] \tag{1.7.3}$$

Introduce currents $T_k(z) = \sum_{r\in\mathbb{Z}} T_k[r]z^{-r}$. Then relations (1.7.2)–(1.7.3) can be rewritten in current form

$$\begin{aligned} &f_{k,n}(w/z)T_1(z)T_k(w) - f_{k,n}(z/w)T_k(w)T_1(z) = \\ &\qquad - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 \Big((k+1)\frac{w}{z}\delta'(\frac{w}{z})T_{k+1}(w) + w\delta(\frac{w}{z})\partial_w T_{k+1}(w)\Big) \end{aligned} \tag{1.7.4}$$

$$\begin{aligned} &f_{n-k,n}(w/z)T_{n-1}(z)T_k(w) - f_{n-k,n}(z/w)T_k(w)T_{n-1}(z) = \\ &\qquad - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 \Big((n-k+1)\frac{w}{z}\delta'(\frac{w}{z})T_{k-1}(w) + w\delta(\frac{w}{z})\partial_w T_{k-1}(w)\Big) \end{aligned} \tag{1.7.5}$$

Also note that $T_0(z) = T_n(z) = 1$.

Remark 1.7.1. There are different approaches to definition of q-W-algebra. For example, in [FF96] algebra $\mathcal{W}_{q,p}(\mathfrak{sl}_n)$ was defined via bosonization. The currents $T_k(z)$ satisfy relation [FF96, Thm. 2]

$$\begin{aligned} &f_{k,n}(w/z)T_1(z)T_k(w) - f_{k,n}(z/w)T_k(w)T_1(z) = \\ &\qquad = \frac{(1-q)(1-p/q)}{1-p}\Big(\delta(w/zp)T_{k+1}(z) - \delta(wp^k/z)T_{k+1}(w)\Big), \end{aligned} \tag{1.7.6}$$

where

$$f_{k,n}(x) = \frac{(x|p^{m-1}q, p^m q^{-1}, p^n, p^{n-1}; p^n)}{(x|p^{m-1}, p^m, p^{n-1}q, p^n q^{-1}; p^n)}. \tag{1.7.7}$$

One can check that limit $p \to 1$ gives relation (1.7.4). However [FF96] do not provide presentation of $\mathcal{W}_{q,p}(\mathfrak{sl}_n)$ in terms of generators and relations.

In the paper [Neg18] relation [Neg18, (2.62)] defines algebra $\mathcal{W}_{q,p}(\mathfrak{gl}_n)$ which (non-essentially) differs from $\mathcal{W}_{q,p}(\mathfrak{sl}_n)$ mentioned above (and from $\mathcal{W}_q(\mathfrak{sl}_n)$ defined above).

Twisted q-W-algebras

Twisted q-W-algebra depends on remainder of n_{tw} modulo n. If $n_{tw} = 0$, then we get definition of non-twisted q-W-algebra from last section. One can find definition of $\mathcal{W}_{q,p}(\mathfrak{sl}_2, 1)$ in [Shi04, (37)–(38)].

Definition 1.7.2. *Algebra $\mathcal{W}_q(\mathfrak{sl}_n, n_{tw})$ is generated by $T_k^{tw}[r]$ for $r \in n_{tw}k/n+\mathbb{Z}$ and $k = 1, \ldots, n-1$. It is convenient to add $T_0^{tw}[r] = T_n^{tw}[r] = \delta_{r,0}$. The defining relations are*

$$\sum_{l=0}^{\infty} f_{k,n}[l]\Big(T_1^{tw}[r-l]T_k^{tw}[s+l] - T_k^{tw}[s-l]T_1^{tw}[r+l]\Big) = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(kr - s)T_{k+1}^{tw}[r+s], \tag{1.7.8}$$

$$\sum_{l=0}^{\infty} f_{n-k,n}[l]\Big(T_{n-1}^{tw}[r-l]T_k^{tw}[s+l] - T_k^{tw}[s-l]T_{n-1}^{tw}[r+l]\Big) = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2\big((n-k)r - s\big)T_{k-1}^{tw}[r+s]. \tag{1.7.9}$$

Let us rewrite relations (1.7.8)–(1.7.9) in the current form. Define currents

$$T_k^{tw}(z) := \sum T_k^{tw}[r]z^{-r}, \quad T_k(z) := z^{\frac{kn_{tw}}{n}}T_k^{tw}(z), \tag{1.7.10}$$

$$T_k^{\circ}(z) := z^{-\frac{(n-k)n_{tw}}{n}}T_k^{tw}(z) = z^{-n_{tw}}T_k(z). \tag{1.7.11}$$

Note that

$$T_0(z) = T_n^{\circ}(z) = 1 \quad T_n(z) = T_0^{\circ}(z) = z^{n_{tw}}. \tag{1.7.12}$$

Proposition 1.7.1. *Relation* (1.7.8) *is equivalent to*

$$\begin{aligned} &f_{k,n}(w/z)T_1(z)T_k(w) - f_{k,n}(z/w)T_k(w)T_1(z) = \\ &\qquad = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2\left((k+1)\frac{w}{z}\delta'(w/z)T_{k+1}(w) + w\delta(w/z)\partial_w T_{k+1}(w)\right). \end{aligned} \tag{1.7.13}$$

Relation (1.7.9) *is equivalent to*

$$\begin{aligned} &f_{n-k,n}(w/z)T_{n-1}^{\circ}(z)T_k^{\circ}(w) - f_{n-k,n}(z/w)T_k^{\circ}(w)T_{n-1}^{\circ}(z) = \\ &\qquad - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2\left((n-k+1)\frac{w}{z}\delta'(w/z)T_{k-1}^{\circ}(w) + w\delta(w/z)\partial_w T_{k-1}^{\circ}(w)\right) \end{aligned} \tag{1.7.14}$$

Remark 1.7.2. In non-twisted case we have relations (1.7.4) and (1.7.5) for currents $T_k(z)$. In twisted case we have the same relations, but for two different sets of currents $T_k(z)$ and $T_k^{\circ}(z)$. One should also keep in mind (1.7.12).

1.7.2 Connection of $\mathcal{W}_q(\mathfrak{sl}_n, n_{tw})$ with $\mathfrak{Diff}_q$

Connection between $\mathcal{W}_q(\mathfrak{sl}_n)$ and $\mathfrak{Diff}_q$ is known (see [FHS+10, Prop. 2.14] or [Neg18, Prop. 2.25]). In this section we generalize it for arbitrary n_{tw}.

Let $\mathfrak{Heis}$ be a Heisenberg algebra generated by $\tilde{H}_j$ with relation $[\tilde{H}_i, \tilde{H}_j] = ni\delta_{i+j,0}$. We will prove that there is a surjective homomorphism $\mathfrak{Diff}_q \twoheadrightarrow \mathcal{W}_q(\mathfrak{sl}_n, n_{tw}) \otimes U(\mathfrak{Heis})$. Secretly, generators H_j are mapped to $\tilde{H}_j$ under the homomorphism. Let us introduce a notation to describe this homomorphism more precisely.

Define

$$\varphi_-(z) = \sum_{j>0} \frac{q^{-j/2} - q^{j/2}}{j}H_{-j}z^j, \qquad \varphi_+(z) = -\sum_{j>0} \frac{q^{j/2} - q^{-j/2}}{j}H_j z^{-j}, \tag{1.7.15}$$

$$\tilde{\varphi}_-(z) = \sum_{j>0} \frac{q^{-j/2} - q^{j/2}}{j}\tilde{H}_{-j}z^j, \qquad \tilde{\varphi}_+(z) = -\sum_{j>0} \frac{q^{j/2} - q^{-j/2}}{j}\tilde{H}_j z^{-j}. \tag{1.7.16}$$

Also, let introduce notation

$$\varphi(z) = \varphi_-(z) + \varphi_+(z), \qquad \tilde{\varphi}(z) = \tilde{\varphi}_-(z) + \tilde{\varphi}_+(z). \tag{1.7.17}$$

Define

$$\tilde{T}_k(z) = \frac{1}{k!}\exp\left(-\frac{k}{n}\varphi_-(z)\right)E^k(z)\exp\left(-\frac{k}{n}\varphi_+(z)\right). \tag{1.7.18}$$

Note that $\tilde{T}_k(z)$ commute with H_j.

Let $J_{\mu,n,n_{tw}}$ be two sided ideal in $\mathfrak{Diff}_q$ generated by $c - n$, $c' - n_{tw}$ and $\tilde{T}_n(z) - \mu^n z^{n_{tw}}$ (here $\mu \in \mathbb{C}\backslash\{0\}$). Parameter μ is not essential since automorphism $E_{a,b} \mapsto \mu^{-a}E_{a,b}$ maps $J_{\mu,n,n_{tw}}$ to $J_{1,n,n_{tw}}$. So we will abbreviate $J_{n,n_{tw}} = J_{\mu,n,n_{tw}}$.

Lemma 1.7.2. *$\tilde{T}_k(z) \in J_{n,n'}$ for $k > n$.*

Proof. It holds in $U(\mathfrak{Diff}_q)/J_{n,n_tw}$

$$E^k(z) = n!\mu^n z^{n_{tw}} E^{k-n}(z) :\exp\varphi(z): \tag{1.7.19}$$

On the other hand,

$$E^{k-n}(z) :\exp\varphi(w):= \frac{(z-w)^{2(k-n)}}{(z-qw)^{k-n}(z-q^{-1}w)^{k-n}}\exp\varphi_-(w)E^{k-n}(z)\exp\varphi_+(w) \tag{1.7.20}$$

Hence, $E^{k-n}(z) :\exp\varphi(z):= 0$. □

Theorem 1.7.1. *There is an algebra isomorphisms $\mathcal{S}\colon \mathcal{W}_q(\mathfrak{sl}_n, n_{tw}) \otimes U(\mathfrak{Heis}) \xrightarrow{\sim} U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$ such that*

$$T_k(z) \mapsto \mu^{-k}\tilde{T}_k(z); \quad \tilde{H}_j \mapsto H_j \tag{1.7.21}$$

The map $\mathcal{P}$ in opposite direction is given by

$$H_j \mapsto \tilde{H}_j; \quad c \mapsto n; \quad c' \mapsto n_{tw}; \tag{1.7.22}$$

$$E(z) \mapsto \ \mu\exp\left(\frac{1}{n}\tilde{\varphi}_-(z)\right)T_1(z)\exp\left(\frac{1}{n}\tilde{\varphi}_+(z)\right) \tag{1.7.23}$$

$$F(z) \mapsto -\frac{\mu^{-1}z^{-n_{tw}}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2}\exp\left(-\frac{1}{n}\tilde{\varphi}_-(z)\right)T_{n-1}(z)\exp\left(-\frac{1}{n}\tilde{\varphi}_+(z)\right) \tag{1.7.24}$$

The rest of this section is devoted to proof of Theorem 1.7.1. First of all, we will prove that formula (1.7.21) indeed defines a homomorphism $\mathcal{S}\colon \mathcal{W}_q(\mathfrak{sl}_n, n_{tw}) \otimes U(\mathfrak{Heis}) \to U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$ (see Proposition 1.7.7). Then we prove that formulas (1.7.22)–(1.7.24) defines a homomorphism in opposite direction (see Proposition 1.7.8). Finally, we note that maps $\mathcal{P}$ and $\mathcal{S}$ are mutually inverse.

Proposition 1.7.3. *Currents $\tilde{T}_k(z)$ (considered as power series with coefficients in $U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$) satisfy relation* (1.7.13).

Proof. Let us define power series in two variables

$$\mathbb{E}^{(k+1)}(z,w) = (z-qw)(z-q^{-1}w)E(z)E^k(w) \tag{1.7.25}$$

According Corollary 1.11.5, $\mathbb{E}^{(k+1)}(z,w)$ is regular in sense of Definition 1.10.2. Following relations follows from results of Section 1.10

$$\mathbb{E}^{(k+1)}(z,w) = (z-qw)(z-q^{-1}w)E^k(w)E(z) \tag{1.7.26}$$

$$\mathbb{E}^{(k+1)}(w,w) = (1-q)(1-q^{-1})w^2E^{k+1}(w) \tag{1.7.27}$$

$$\partial_z\mathbb{E}^{(k+1)}(z,w)\Big|_{z=w} = (1-q)(1-q^{-1})w^2\frac{1}{k+1}\partial_w E^{k+1}(w) + (1-q)(1-q^{-1})wE^{k+1}(w) \tag{1.7.28}$$

More precisely, (1.7.26)–(1.7.27) easily follows from Propositions 1.10.2. One can find a proof of (1.7.28) at the end of Section 1.10.

It is straightforward to check that

$$f_{k,n}(w/z)\tilde{T}_1(z)\tilde{T}_k(w) = \frac{(1-q\frac{w}{z})(1-q^{-1}\frac{w}{z})}{k!(1-\frac{w}{z})^2}\exp\left(-\frac{1}{n}\left(\varphi_-(z)+k\varphi_-(w)\right)\right)$$
$$E(z)E^k(w)\ \exp\left(-\frac{1}{n}\left(\varphi_+(z)+k\varphi_+(w)\right)\right) \tag{1.7.29}$$

Formulas (1.7.29) and (1.7.26)–(1.7.28) implies that

$$f_{k,n}(w/z)\tilde{T}_1(z)\tilde{T}_k(w) - f_{k,n}(z/w)\tilde{T}_k(w)\tilde{T}_1(z) =$$
$$\frac{1}{k!}\exp\left(-\frac{1}{n}\left(\varphi_-(z)+k\varphi_-(w)\right)\right)\ \mathbb{E}^{(k+1)}(z,w)\ \exp\left(-\frac{1}{n}\left(\varphi_+(z)+k\varphi_+(w)\right)\right)\ \partial_w\left(w^{-1}\delta(w/z)\right) =$$
$$= (1-q)(1-q^{-1})(k+1)\tilde{T}_{k+1}(w)\frac{w}{z}\delta'(w/z) + (1-q)(1-q^{-1})w\tilde{T}'_{k+1}(w)\delta(w/z)$$

□

Lemma 1.7.4. *The following OPE holds in $\mathfrak{Diff}_q$*

$$F(z)E^k(w) = k(c-k+1)\frac{\frac{w}{z}E^{k-1}(w)}{(1-\frac{w}{z})^2} + k\frac{w\partial_w E^{k-1}(w) - w:\varphi'(w)E^{k-1}(w): - c'E^{k-1}(w)}{1-\frac{w}{z}} + \text{reg}; \tag{1.7.30}$$

or, equivalently

$$[F(z),E^k(w)] = k(c-k+1)E^{k-1}(w)\frac{w}{z}\delta'(w/z)+$$
$$k\left(w\partial_w E^{k-1}(w) - w:\varphi'(w)E^{k-1}(w): - c'E^{k-1}(w)\right)\delta(w/z). \tag{1.7.31}$$

Proof. Denote by $E(\underline{w}) = E(w_1)\cdots E(w_k)$. We will write $:F(z)E(\underline{w}): = F_+(z)E(\underline{w}) + E(\underline{w})F_-(z)$; this definition reminds standard Definition 1.10.3, but is applied in different situation ($E(\underline{w})$ is not a current in one variable). Note that

$$F(z)E(\underline{w}) = [F_-(z),E(\underline{w})] + :F(z)E(\underline{w}): = \sum_j E(w_1)\cdots E(w_{j-1})$$
$$\left(\frac{H(q^{\frac12}w_j) - H(q^{-\frac12}w_j) - c'}{1-\frac{w_j}{z}} + c\frac{w_j}{z}\frac{1}{(1-\frac{w_j}{z})^2}\right)E(w_{j+1})\cdots E(w_k) + :F(z)E(\underline{w}): \tag{1.7.32}$$

Recall that $w\varphi'(w) = H(q^{-\frac{1}{2}}w) - H(q^{\frac{1}{2}}w)$. It follows from (1.2.9) that

$$\left(H(q^{\frac{1}{2}}w_j) - H(q^{-\frac{1}{2}}w_j)\right) E(w_l) = -\frac{2E(w_l)}{1-\frac{w_l}{w_j}} + \frac{E(w_l)}{1-q\frac{w_l}{w_j}} + \frac{E(w_l)}{1-q^{-1}\frac{w_l}{w_j}} - w_j :\varphi'(w_j)E(w_l):, \quad (1.7.33)$$

$$E(w_j)\left(H(q^{\frac{1}{2}}w_l) - H(q^{-\frac{1}{2}}w_l)\right) = \frac{2E(w_j)}{1-\frac{w_l}{w_j}} - \frac{E(w_j)}{1-q\frac{w_l}{w_j}} - \frac{E(w_j)}{1-q^{-1}\frac{w_l}{w_j}} - w_l :\varphi'(w_l)E(w_j):. \quad (1.7.34)$$

Using identity

$$\frac{1}{(1-\frac{w_j}{z})(1-\frac{w_j}{w_l})} - \frac{1}{(1-\frac{w_l}{z})(1-\frac{w_j}{w_l})} = -\frac{\frac{w_l}{z}}{(1-\frac{w_l}{z})(1-\frac{w_j}{z})} \quad (1.7.35)$$

we obtain

$$\left.\frac{E(w_l)}{(1-\frac{w_j}{z})(1-\frac{w_j}{w_l})} - \frac{E(w_j)}{(1-\frac{w_l}{z})(1-\frac{w_j}{w_l})}\right|_{w=w_j=w_l} = -\frac{\frac{w}{z}E(w)}{(1-\frac{w}{z})^2} + \frac{w\partial_w E(w)}{1-\frac{w}{z}}. \quad (1.7.36)$$

Finally, we conclude that

$$\left.\sum_j E(w_1)\cdots E(w_{j-1})\,\frac{H(q^{\frac{1}{2}}w_j) - H(q^{-\frac{1}{2}}w_j)}{1-\frac{w_j}{z}}\,E(w_{j+1})\cdots E(w_k)\right|_{w=w_1=\cdots=w_k} =$$

$$-\binom{k}{2}\frac{2\frac{w}{z}E^{k-1}(w)}{(1-\frac{w}{z})^2} + \binom{k}{2}\frac{2wE^{k-2}(w)\,\partial_w E(w)}{1-\frac{w}{z}} - k\frac{w:\varphi'(w)E^{k-1}(w):}{1-\frac{w}{z}} \quad (1.7.37)$$

Relation (1.7.30) follows from (1.7.32) and (1.7.37) □

Proposition 1.7.5. *In algebra $U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$ holds*

$$F(z) = -\frac{\mu^{-n}z^{-n_{tw}}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}\exp\left(-\frac{1}{n}\varphi_-(z)\right)\tilde{T}_{n-1}(z)\exp\left(-\frac{1}{n}\varphi_+(z)\right) \quad (1.7.38)$$

Proof. Usign relation $\tilde{T}_n(z) - \mu^n z^{n_{tw}} \in J_{n,n_{tw}}$, we find a relation in $U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$

$$F(z)E^n(w) = n!\mu^n w^{n_{tw}} F(z) :\exp(\phi(w)): =$$
$$n!\mu^n w^{n_{tw}}\frac{(1-qw/z)(1-q^{-1}w/z)}{(1-w/z)^2}\exp(\phi_-(w))F(z)\exp(\phi_+(w)) \quad (1.7.39)$$

Comparing coefficient of $(z-w)^{-2}$ with (1.7.30), we obtain

$$nE^{n-1}(w) = n!\mu^n w^{n_{tw}}(1-q)(1-q^{-1})\exp(\phi_-(w))F(w)\exp(\phi_+(w)) \quad (1.7.40)$$

□

Denote by $\tilde{T}^\circ_k(z) = \mu^{-n}z^{-n_{tw}}\tilde{T}_k(z)$.

Proposition 1.7.6. *Following relation holds in $U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$*

$$f_{n-k,n}(w/z)\tilde{T}^\circ_{n-1}(z)\tilde{T}^\circ_k(w) - f_{n-k,n}(z/w)\tilde{T}^\circ_k(w)\tilde{T}^\circ_{n-1}(z) =$$
$$-(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2\left((n-k+1)\frac{w}{z}\delta'(\frac{w}{z})\tilde{T}^\circ_{k-1}(w) + w\delta(\frac{w}{z})\partial_w\tilde{T}^\circ_{k-1}(w)\right) \quad (1.7.41)$$

Proof. Proposition 1.7.5 imply

$$f_{n-k,n}(w/z)\tilde{T}^{\circ}_{n-1}(z)\tilde{T}^{\circ}_{k}(w) - f_{n-k,n}(z/w)\tilde{T}^{\circ}_{k}(w)\tilde{T}^{\circ}_{n-1}(z) =$$
$$-(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2\frac{\mu^{-n}w^{-n_{tw}}}{k!}\exp\left(\frac{1}{n}\varphi_-(z)-\frac{k}{n}\varphi_-(w)\right)[F(z),E^k(w)]\exp\left(\frac{1}{n}\varphi_+(z)-\frac{k}{n}\varphi_+(w)\right).$$

It is straightforward to finish the proof using (1.7.31). □

Proposition 1.7.7. *Formula* (1.7.21) *defines a homomorphism $\mathcal{S}$ from $\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})\otimes U(\mathfrak{Heis})$ to the algebra $U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$.*

Proof. Evidently, H_j and $\tilde{T}_k(z)$ commute, and H_j form a Heisenberg algebra. We only have to check that $\frac{1}{\mu^k}\tilde{T}_k(z)$ form $\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})$ algebra. Relation of $\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})$ algebra follows from Propositions 1.7.3 and 1.7.6. □

Proposition 1.7.8. *Formulas* (1.7.22)–(1.7.24) *defines a homomorphism $\mathcal{P}$ from $U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$ to the algebra $\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})\otimes U(\mathfrak{Heis})$.*

Proof. Let us check that these formulas define morphism from $\mathfrak{Diff}_q$. According to Proposition 1.2.2, it is enough to prove relations (1.2.8)–(1.2.13). It is done in Section 1.12. Evidently, $\mathcal{P}$ annihilates $J_{n,n_{tw}}$. □

Proposition 1.7.9. *Maps $\mathcal{P}$ and $\mathcal{S}$ are mutually inverse.*

Proof. Let us prove $\mathcal{PS}=\mathrm{id}_{\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})\otimes U(\mathfrak{Heis})}$ first. The algebra $\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})$ is generated by modes of $T_1(z)$. Hence it is sufficient to check $\mathcal{PS}(\tilde{H}_n)=\tilde{H}_n$ and $\mathcal{PS}\big(T_1(z)\big)=T_1(z)$. Both of them are straightforward.

The algebra $U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$ is generated by modes of $E(z)$ and $F(z)$. Evidently, $\mathcal{SP}\big(E(z)\big)=E(z)$. Proposition 1.7.5 implies $\mathcal{SP}\big(F(z)\big)=F(z)$. □

Proof of Theorem 1.7.1. Follows from Proposition 1.7.7, 1.7.8 and 1.7.9 □

1.7.3 Bosonization of $\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})$

Let σ be as in (1.4.1). Corollary 1.3.10 states that representation $\mathcal{F}^{\sigma}_u$ actually does not depend on m' and m; it is determined by n' and n. Let us denote the representation by $\mathcal{F}^{(n',n)}_u$.

Fock representation via $\mathfrak{Diff}_q$

In this section we will discuss connection of twisted q-W algebras and twisted representations $\mathcal{F}^{(n',n)}_u$.

Lemma 1.7.10. *In representation $\mathcal{F}^{(n',n)}_u$ operator $\tilde{T}_n(z)$ acts by $-q^{-1/2}uz^{n'}\frac{1}{(q^{1/2}-q^{-1/2})^n}$.*

Proof. We will use formula (1.4.11) to calculate $E^n(z)$.

By Proposition 1.10.2

$$\psi_{(a)}(q^{-1/2}z)\psi_{(b)}(q^{-1/2}z) = -\psi_{(b)}(q^{-1/2}z)\psi_{(a)}(q^{-1/2}z) \tag{1.7.42}$$
$$\psi_{(a)}(q^{-1/2}z)\psi^*_{(b)}(q^{1/2}z) = -\psi^*_{(b)}(q^{1/2}z)\psi_{(a)}(q^{-1/2}z) \tag{1.7.43}$$

for any a,b (even for $a=b$).

Consider a sequence of numbers $0\leqslant a_1,\dots,a_n\leqslant n-1$ such that $a_{i+1}-a_i\equiv -n' \mod n$. Thus,

$$E^n(z) = n!uq^{-1/2}z^{n'+n}\psi_{(a_1)}(q^{-1/2}z)\psi^*_{(a_2)}(q^{1/2}z)\psi_{(a_2)}(q^{-1/2}z)\psi^*_{(a_3)}(q^{1/2}z)\cdots\psi_{(a_n)}(q^{-1/2}z)\psi^*_{(a_1)}(q^{1/2}z) \tag{1.7.44}$$

Using bosonization (1.5.31), (1.5.32) we obtain (cf. (1.7.15) and (1.7.17))

$$E^n(z) = -n! u q^{-1/2} z^{n'} \frac{1}{(q^{1/2}-q^{-1/2})^n} :\exp(\varphi(z)): \tag{1.7.45}$$

Consequently,

$$\tilde{T}_n(z) = -q^{-1/2} u z^{n'} \frac{1}{(q^{1/2}-q^{-1/2})^n} \tag{1.7.46}$$

□

Proposition 1.7.11. *Suppose, M_i are representation of $\mathfrak{Diff}_q$ such that ideal $J_{\mu_i,n_i,n_i'}$ acts by zero (for $i=1,\dots,k$). Then $J_{\mu,n,n'}$ acts by zero on $M_1\otimes\dots\otimes M_k$ for $n=\sum_{i=1}^k n_i$, $n'=\sum_{i=1}^k n_i'$ and $\mu^n=\mu_1^{n_1}\cdots\mu_k^{n_k}$.*

Proof. Recall that $E^{n_i+1}(z)\in J_{n_i,n_i'}$ by Lemma 1.7.2. Thus, $E^n(z)$ act on $M_1\otimes\dots\otimes M_k$ as

$$\frac{n!}{n_1!\cdots n_k!}E^{n_1}(z)\otimes\dots\otimes E^{n_k}(z) = n!\prod_{i=1}^k \mu_i^{n_i} z^{n_i'} :\exp(\varphi(z)):. \tag{1.7.47}$$

□

Proposition 1.7.12. *Ideal $J_{\mu,nd,n'd}$ acts by zero on $\mathcal{F}_{u_1}^{(n',n)}\otimes\dots\otimes\mathcal{F}_{u_d}^{(n',n)}$ for*

$$\mu = (-1)^{\frac{1}{n}} \frac{q^{-\frac{1}{2n}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} (u_1\cdots u_d)^{\frac{1}{nd}}, \tag{1.7.48}$$

Proof. Follows from Lemma 1.7.10 and Proposition 1.7.11. □

Theorem 1.7.2. *There is an action of $\mathcal{W}(\mathfrak{sl}_{nd},n'd)\otimes U(\mathfrak{Heis})$ on $\mathcal{F}_{u_1}^{(n',n)}\otimes\dots\otimes\mathcal{F}_{u_d}^{(n',n)}$ such that action of $\mathfrak{Diff}_q$ factors through $\mathcal{W}(\mathfrak{sl}_{nd},n'd)\otimes U(\mathfrak{Heis})$.*

Proof. According to Proposition 1.7.12, algebra $U(\mathfrak{Diff}_q)/J_{nd,n'd}$ acts on $\mathcal{F}_{u_1}^{(n',n)}\otimes\dots\otimes\mathcal{F}_{u_d}^{(n',n)}$. By Theorem 1.7.1, algebra $U(\mathfrak{Diff}_q)/J_{nd,n'd}$ is isomorphic to $\mathcal{W}(\mathfrak{sl}_{nd},n'd)\otimes U(\mathfrak{Heis})$. □

Remark 1.7.3. One can consider tensor product of Fock modules with different twists $\mathcal{F}_{u_1}^{\sigma_1}\otimes\dots\otimes\mathcal{F}_{u_d}^{\sigma_d}$. According to Proposition 1.7.11 algebra $\mathcal{W}(\mathfrak{sl}_{\sum n_i},\sum n_i')\otimes U(\mathfrak{Heis})$ acts on this space. Obtained representation is 'irregular' (cf. [Nag15]). In Section 1.13.2 we consider an intertwiner between irregular and (graded completion of) regular representation.

Explicit formula for bosonization

Below we will write explicit formula for bosonization of $\mathcal{W}_q(\mathfrak{sl}_{nd},n'd)$. This bosonization comes from action of $\mathcal{W}(\mathfrak{sl}_{nd},n'd)$ on $\mathcal{F}_{u_1}^{(n',n)}\otimes\dots\otimes\mathcal{F}_{u_d}^{(n',n)}$. Recall that realization of $\mathcal{F}_u^{(n',n)}$ is written via $a_b[k]$ for $b=0,\dots n-1$. Denote by $a_b^i[k]$ (for $i=1,\dots,d$ and $b=0,\dots,n-1$) generators of Heisenberg algebra action on ith factor of the tensor product $\mathcal{F}_{u_1}^{(n',n)}\otimes\dots\otimes\mathcal{F}_{u_d}^{(n',n)}$.

To write action of $\mathcal{W}_q(\mathfrak{sl}_{nd},n'd)$ we need to introduce slightly different version of Heisenberg algebra. Namely, consider an algebra, generated by $\eta_b^i[k]$ for $b=0,\dots n-1$; $i=1,\dots,d$ and $k\in\mathbb{Z}$. Relations are given by linear dependence and commutation relations

$$\sum_{i=1}^d\sum_{b=0}^{n-1}\eta_b^i[k]=0 \tag{1.7.49}$$

$$\left[\eta_{b_1}^{i_1}[k_1],\eta_{b_2}^{i_2}[k_2]\right] = k_1\delta_{k_1+k_2,0}\left(\delta_{i_1,i_2}\delta_{b_1,b_2}-\frac{1}{nd}\right) \tag{1.7.50}$$

Let us define representation $F^\eta \otimes \mathbb{C}[\mathbf{Q}^d_{(n)}]$ (cf. Section 1.4.2). Lattice $\mathbf{Q}^d_{(n)}$ consist of elements $\sum \lambda^i_b Q^i_b$ such that $\lambda^i_b \in \mathbb{Z}$ and for any i it holds $\sum_b \lambda^i_b = 0$. Define an action

$$\eta^i_b[0]\, e^{\sum \lambda^i_b Q^i_b} = \lambda^i_b\, e^{\sum \lambda^i_b Q^i_b} \tag{1.7.51}$$

First factor F^η is a Fock space for subalgebra $\eta^i_b[k]$ for $k \neq 0$. We can consider $F^\eta \otimes \mathbb{C}[\mathbf{Q}^d_{(n)}]$ as representation of whole Heisenberg algebra as follows: $\eta^i_b[k]$ for $k \neq 0$ acts on first factor, $\eta^i_a[0]$ acts on the second factor by (1.7.51). Note, that also $\mathbb{C}[\mathbf{Q}^d_{(n)}]$ acts on $F^\eta \otimes \mathbb{C}[\mathbf{Q}^d_{(n)}]$. Let us introduce notation

$$\eta^i_b(z) = \sum_{k\neq 0} \frac{1}{k}\eta^i_b[k] z^{-k} + Q^i_b + \eta^i_b[0]\log z \tag{1.7.52}$$

Proposition 1.7.13. *There is an action of $\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)$ on $F^\eta \otimes \mathbb{C}[\mathbf{Q}^d_{(n)}]$ given by formulas*

$$\begin{aligned} T_1(z) \mapsto (-1)^{-\frac{1}{n}}(u_1\cdots u_d)^{-\frac{1}{nd}}(q^{\frac12}-q^{-\frac12})q^{\frac{1}{2n}} \sum_{i=1}^d \sum_{b-a\equiv -n' \bmod n} \\ u_i^{\frac1n} q^{\frac{a+b-n}{2n}} z^{\frac{n'-a+b}{n}+1} :\exp\Big(\eta^i_b(q^{1/2}z) - \eta^i_a(q^{-1/2}z)\Big): \epsilon^{(i)}_{a,b} \end{aligned} \tag{1.7.53}$$

$$\begin{aligned} T_{nd-1}(z) \mapsto -(-1)^{\frac{1}{n}}(u_1\cdots u_d)^{\frac{1}{nd}}(q^{\frac12}-q^{-\frac12})q^{-\frac{1}{2n}} z^{n'd} \sum_{i=1}^d \sum_{b-a\equiv n' \bmod n} \\ u_i^{-\frac1n} q^{\frac{-a-b+n}{2n}} z^{\frac{-n'-a+b}{n}+1} :\exp\Big(\eta^i_b(q^{-1/2}z) - \eta^i_a(q^{1/2}z)\Big): \epsilon^{(i)}_{a,b} \end{aligned} \tag{1.7.54}$$

here $\epsilon^{(i)}_{a,b} = \prod_r (-1)^{\eta^i_r[0]}$ (we consider product over such r that $a-1 \geqslant r \geqslant b$ for $a > b$ and $b-1 \geqslant r \geqslant a$ for $b > a$).

Proof. Denote by $\Big[\mathcal{F}^{(n',n)}_{u_1} \otimes \cdots \otimes \mathcal{F}^{(n',n)}_{u_d}\Big]_h$ subspace of such $v \in \mathcal{F}^{(n',n)}_{u_1} \otimes \cdots \otimes \mathcal{F}^{(n',n)}_{u_d}$ that $H_k v = 0$ for $k > 0$. According to Theorem 1.7.2, the algebra $\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)$ acts on $\Big[\mathcal{F}^{(n',n)}_{u_1} \otimes \cdots \otimes \mathcal{F}^{(n',n)}_{u_d}\Big]_h$.

On the other hand map $\eta^i_b[k] \mapsto a^i_b[k] - \sum_{b,i} a^i_b[k]$ is a homomorphism. Therefore, one can identify $F^\eta \otimes \mathbb{C}[\mathbf{Q}^d_{(n)}]$ and $\Big[\mathcal{F}^{(n',n)}_{u_1} \otimes \cdots \otimes \mathcal{F}^{(n',n)}_{u_d}\Big]_h$.

Substitution of (1.4.17), (1.4.18), and (1.7.48) to (1.7.23), (1.7.24) finishes the proof. □

Denote obtained representation by $\mathcal{F}^{\mathcal{W}_q(\mathfrak{sl}_{nd},n'd)}_{u_1,\dots,u_d}$.

Remark 1.7.4. The parameter μ is determined by $u_1,\dots,u_d$ only up to nd-th root of unity. The modules $\mathcal{F}^{\mathcal{W}_q(\mathfrak{sl}_{nd},n'd)}_{u_1,\dots,u_d}$ with different μ are non-isomorphic in general (so notation $\mathcal{F}^{\mathcal{W}_q(\mathfrak{sl}_{nd},n'd)}_{u_1,\dots,u_d}$ is ambiguous). For example, one can see this from the highest weights λ_s defined in the next section.

The modules $\mathcal{F}^{\mathcal{W}_q(\mathfrak{sl}_{nd},n'd)}_{u_1,\dots,u_d}$ with different μ are related by an external automorphism of $\mathcal{W}_q(\mathfrak{sl}_n, n_{tw})$.

Example 1.7.1. Let us consider case of twisted Virasoro algebra i.e. $n = 2$, $n' = 1$, $d = 1$. Then everything is expressed via one boson $\eta(z)$ with relation $[\eta[k_1], \eta[k_2]] = \frac12 k_1\delta_{k_1+k_2,0}$ and $[\eta[0], Q] = \frac12$. So there is an action of $\mathcal{W}_q(\mathfrak{sl}_2, 1)$ on $F^\eta \otimes \mathbb{C}[\mathbb{Z}]$ given by

$$\begin{aligned} T_1(z) \mapsto (-1)^{-\frac12}(q^{\frac12}-q^{-\frac12})(-1)^{\eta[0]} \\ \Big[z :\exp\Big(\eta(q^{1/2}z)+\eta(q^{-1/2}z)\Big): + z^2 :\exp\Big(-\eta(q^{1/2}z)-\eta(q^{-1/2}z)\Big):\Big] \end{aligned} \tag{1.7.55}$$

We can simplify the formula using conjugation by $(-1)^{\eta[0]^2/2}$

$$T_1(z) \mapsto (q^{\frac12}-q^{-\frac12})\Big[z :\exp\Big(\eta(q^{1/2}z)+\eta(q^{-1/2}z)\Big): + z^2 :\exp\Big(-\eta(q^{1/2}z)-\eta(q^{-1/2}z)\Big):\Big] \tag{1.7.56}$$

Explicit formulas for strange bosonization

To write formulas for strange bosonization we need to consider Heisenberg algebra generated by $\xi_i[k]$ for $i = 1, \dots d$ and $k \in \mathbb{Z}$. Relations are given by linear dependence and commutation relations

$$\sum_{i=1}^{d} \xi_i[nk] = 0 \tag{1.7.57}$$

$$[\xi_{i_1}[k_1], \xi_{i_2}[k_2]] = k_1 \delta_{k_1+k_2,0}\, \delta_{i_1,i_2} \qquad \text{for either } n \nmid k_1 \text{ or } n \nmid k_2 \tag{1.7.58}$$

$$[\xi_{i_1}[nj_1], \xi_{i_2}[nj_2]] = nj_1 \delta_{j_1+j_2,0} \left(\delta_{i_1,i_2} - \frac{1}{d} \right) \tag{1.7.59}$$

Denote corresponding Fock module by F^{ξ}.

Proposition 1.7.14. *There is an action of $\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)$ on F^{ξ} given by*

$$T_1(z) \mapsto -(-1)^{-\frac{1}{n}} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{n(q^{\frac{1}{2n}} - q^{-\frac{1}{2n}})} (u_1 \cdots u_d)^{-\frac{1}{nd}} z^{\frac{n'}{n}} \sum_{i=1}^{d} u_i^{1/n} \sum_{l=0}^{n-1} \zeta^{ln'} :\exp\left(\sum_k \frac{q^{-k/2n} - q^{k/2n}}{k} \xi_i[k] \zeta^{-kl} z^{-k/n} \right): \tag{1.7.60}$$

$$T_{nd-1}(z) \mapsto -(-1)^{\frac{1}{n}} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{n(q^{\frac{1}{2n}} - q^{-\frac{1}{2n}})} (u_1 \cdots u_d)^{\frac{1}{nd}} z^{\frac{nd-1}{n} n'} \sum_{i=1}^{d} u_i^{-1/n} \sum_{l=0}^{n-1} \zeta^{-ln'} :\exp\left(\sum_k \frac{q^{k/2n} - q^{-k/2n}}{k} \xi_i[k] \zeta^{-kl} z^{-k/n} \right): \tag{1.7.61}$$

Obtained representation is isomorphic to $\mathcal{F}_{u_1,\dots,u_d}^{\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)}$.

Proof. The proof is analogous to proof of Proposition 1.7.13. The only difference is that we have to use (1.4.20), (1.4.21) instead of (1.4.17), (1.4.18). This representation is isomorphic to $\mathcal{F}_{u_1,\dots,u_d}^{\mathcal{W}_q(\mathfrak{sl}_{nd}, n'd)}$ since it also corresponds to $\mathcal{F}_{u_1}^{(n',n)} \otimes \cdots \otimes \mathcal{F}_{u_d}^{(n',n)}$. $\square$

1.7.4 Verma modules vs Fock modules

Connection of Fock module and Verma module is known in non-twisted case. In this Subsection we will generalize it for $\mathcal{W}_q(\mathfrak{sl}_n, n_{tw})$. Denote $d = \gcd(n_{tw}, n)$.

Definition 1.7.3. $\mathcal{W}_q(\mathfrak{gl}_n, n_{tw}) = U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$.

Denote modes of $E^k(z)$ by $E^k[j]$. Namely, $E^k(z) = \sum_d E^k[j] z^{-j}$.

Definition 1.7.4. *Verma module $\mathcal{V}_{\lambda_1,\dots,\lambda_d}^{\mathcal{W}_q(\mathfrak{gl}_n, n_{tw})}$ is a module over $\mathcal{W}_q(\mathfrak{gl}_n, n_{tw})$ with cyclic vector $|\bar{\lambda}\rangle_{\mathfrak{gl}}$ and relations*

$$H_k |\bar{\lambda}\rangle_{\mathfrak{gl}} = 0 \qquad \textit{for } k > 0 \tag{1.7.62}$$

$$E^k[j] |\bar{\lambda}\rangle_{\mathfrak{gl}} = 0 \qquad \textit{for } j + \frac{kn_{tw}}{n} > 0 \tag{1.7.63}$$

$$E^{\frac{ns}{d}}[-sn_{tw}/d] |\bar{\lambda}\rangle_{\mathfrak{gl}} = \frac{ns}{d}!\, \lambda_s |\bar{\lambda}\rangle_{\mathfrak{gl}} \tag{1.7.64}$$

Consider a grading on $\mathcal{W}_q(\mathfrak{gl}_n, n_{tw})$ given by $\deg E_k[j] = -j - \frac{kn_{tw}}{n}$. Verma module $\mathcal{V}^{\mathcal{W}_q(\mathfrak{gl}_n,n_{tw})}_{\lambda_1,\dots,\lambda_d}$ is a graded module with grading defined by requirement $\deg \left|\bar{\lambda}\right\rangle_{\mathfrak{gl}} = 0$.

Proposition 1.7.15. *$\mathcal{V}^{\mathcal{W}_q(\mathfrak{gl}_n,n_{tw})}_{\lambda_1,\dots,\lambda_d}$ is spanned by $E^{k_1}[j_1]\dots E^{k_t}[j_t]\left|\bar{\lambda}\right\rangle_{\mathfrak{gl}}$ for $\frac{j_1}{k_1} \leqslant \frac{j_2}{k_2} \leqslant \dots \leqslant \frac{j_t}{k_t} < -\frac{n_{tw}}{n}$ and $1 \leqslant k_i \leqslant n$.*

Proof. Recall that $E^n(z) = n! z^{n_{tw}} :\exp(\varphi(z)):$. Therefore $E^n[j]$ acts on F^H.

Lemma 1.7.16. *Module F^H is spanned by $E^n[-j_1]\dots E^n[-j_k]|\varnothing\rangle$*

Sketch of the proof. Let us consider operator $E^n_-[-j]$ defined by $\sum_j E^n_-[-j]z^j = \exp(\varphi_-(z))$. On one hand $E^n_-[-j_1]\dots E^n_-[-j_k]|\varnothing\rangle$ is a basis of F^H (this basis coincide with a basis of complete homogeneous polynomials up to renormalization of Heisenberg algebra generators). One the other hand,

$$E^n[-j_1]\dots E^n[-j_k]|\varnothing\rangle = E^n_-[-j_1]\dots E^n_-[-j_k]|\varnothing\rangle + \text{lower terms}, \tag{1.7.65}$$

here lower terms are taken with respect to lexicographical order. □

Remark 1.7.5. Lemma 1.7.16 holds for any exponent $:\exp\left(\sum \alpha_j H_j z^{-j}\right):$ such that $\alpha_{-i} \neq 0$ for $i > 0$. The proof does not use any other properties of coefficients α_j. One can find the proof as the last part of proof of [Neg18, Prop. 2.29].

Define $\mathfrak{Diff}_q^{\geqslant 0}$ and $\mathfrak{Diff}_q^{>0}$ as subalgebras of $\mathfrak{Diff}_q$ spanned by $E_{k,j}$ with $k \geqslant 0$ and $k > 0$ correspondingly.

Lemma 1.7.17. *Vector $\left|\bar{\lambda}\right\rangle_{\mathfrak{gl}} \in \mathcal{V}^{\mathcal{W}_q(\mathfrak{gl}_n,n_{tw})}_{\lambda_1,\dots,\lambda_d}$ is cyclic with respect to action of $\mathfrak{Diff}_q^{>0}$.*

Proof. Theorem 1.7.1 implies that the natural map $U(\mathfrak{Diff}_q^{\geqslant 0}) \to U(\mathfrak{Diff}_q)/J_{n,n_{tw}}$ is surjective. Hence Verma module is generated by non-commutative monomials in E_i and H_j applied to $\left|\bar{\lambda}\right\rangle_{\mathfrak{gl}}$. Using relation $[H_j, E_i] = (q^{-j/2} - q^{j/2})E_{i+j}$, we see that the module is spaned by $E_{i_1}\dots E_{i_k} H_{j_1}\dots H_{j_l}\left|\bar{\lambda}\right\rangle_{\mathfrak{gl}}$. Lemma 1.7.16 implies that the module is spanned by $E_{i_1}\dots E_{i_k} E^n[j_1]\dots E^n[j_l]\left|\bar{\lambda}\right\rangle_{\mathfrak{gl}}$ □

In [Neg17, (3.48)] author states, that algebra $\mathfrak{Diff}_q^{>0}$ has a PBW-like basis $E^{k_1}[j_1]\dots E^{k_t}[j_t]$ for $\frac{j_1}{k_1} \leqslant \frac{j_2}{k_2} \leqslant \dots \leqslant \frac{j_t}{k_t}$. Thus $\mathcal{V}^{\mathcal{W}_q(\mathfrak{gl}_n,n_{tw})}_{\lambda_1,\dots,\lambda_d}$ is spanned by $E^{k_1}[j_1]\dots E^{k_t}[j_t]\left|\bar{\lambda}\right\rangle_{\mathfrak{gl}}$ (with the same condition on k_i and j_i).

Note that if $\frac{j_t}{k_t} > -\frac{n_{tw}}{n}$ then such vector is 0; moreover if $\frac{j_t}{k_t} = -\frac{n_{tw}}{n}$ then $E^{k_t}[j_t]$ acts by multiplication on a constant, hence it can be excluded. Also note that $E^k[j] \in J_{n,n_{tw}}$ for $k \geqslant n+1$, therefore we assume $k_i \leqslant n$. The Proposition 1.7.15 is proven. □

Corollary 1.7.18. *Coefficients of $\operatorname{ch} \mathcal{V}^{\mathcal{W}_q(\mathfrak{gl}_n,n_{tw})}_{\lambda_1,\dots,\lambda_d}$ are less or equal to coefficients of $\prod_{k=1}^{\infty}\left(1 - \mathfrak{q}^{\frac{kd}{n}}\right)^{-d}$.*

Theorem 1.7.3. *If $\mathcal{F}^{(n_{tw}/d,n/d)}_{u_1} \otimes \dots \otimes \mathcal{F}^{(n_{tw}/d,n/d)}_{u_d}$ is irreducible then natural map $p\colon \mathcal{V}^{\mathcal{W}_q(\mathfrak{gl}_n,n_{tw})}_{\lambda_1,\dots,\lambda_d} \to \mathcal{F}^{(n_{tw}/d,n/d)}_{u_1} \otimes \dots \otimes \mathcal{F}^{(n_{tw}/d,n/d)}_{u_d}$ is an isomorphism for*

$$\lambda_s = \left(-q^{-\frac{1}{2}}\right)^s \frac{1}{\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{\frac{ns}{d}}} e_s(u_1, \dots, u_d). \tag{1.7.66}$$

Proof. Let us first prove existence of the map p. This will follow automatically from the following lemma.

Lemma 1.7.19. *The highest vector $|\bar{u}\rangle \in \mathcal{F}_{u_1}^{(n_{tw}/d,n/d)} \otimes \dots \otimes \mathcal{F}_{u_d}^{(n_{tw}/d,n/d)}$ satisfy following conditions*

$$H_k\,|\bar{u}\rangle = 0 \qquad \text{for } k>0 \tag{1.7.67}$$

$$E^k[j]\,|\bar{u}\rangle = 0 \qquad \text{for } j+\frac{kn_{tw}}{n}>0 \tag{1.7.68}$$

$$E^{\frac{ns}{d}}[-sn_{tw}/d]\,|\bar{u}\rangle = \frac{ns}{d}!\,\lambda_s\,|\bar{u}\rangle \tag{1.7.69}$$

Proof. Assertions (1.7.67)–(1.7.68) are evident. Let us check (1.7.69). Denote by $\varepsilon_j(z)$ action of $E(z)$ on jth tensor multiple; in particular $E(z)\mapsto \varepsilon_1(z)+\dots+\varepsilon_d(z)$. Recall that $\varepsilon_j(z)^{\frac{n}{d}+1}=0$. Thus

$$E^{\frac{ns}{d}}(z) \mapsto \frac{\frac{ns}{d}!}{(\frac{n}{d}!)^s} \sum_{k_1<k_2<\dots<k_j} \varepsilon_{k_1}^{\frac{n}{d}}(z)\varepsilon_{k_2}^{\frac{n}{d}}(z)\dots\varepsilon_{k_s}^{\frac{n}{d}}(z)+\dots, \tag{1.7.70}$$

here dots denote summands with a power which is not a multiple of $\frac{n}{d}$ (thus, this summands do not contribute to $E^{\frac{ns}{d}}[-sn_{tw}/d]\,|\bar{u}\rangle$). To finish the proof we note that

$$\frac{\frac{ns}{d}!}{(\frac{n}{d}!)^s}\langle\bar{u}|\,\varepsilon_{k_1}^{\frac{n}{d}}(z)\varepsilon_{k_2}^{\frac{n}{d}}(z)\dots\varepsilon_{k_s}^{\frac{n}{d}}(z)\,|\bar{u}\rangle = \frac{ns}{d}!\,u_{k_1}u_{k_2}\dots u_{k_s}\left(-q^{-1/2}\right)^s z^{\frac{n_{tw}}{d}s}\frac{1}{(q^{1/2}-q^{-1/2})^{\frac{ns}{d}}}. \tag{1.7.71}$$

□

Image of p is whole $\mathcal{F}_{u_1}^{(n_{tw}/d,n/d)}\otimes\dots\otimes\mathcal{F}_{u_d}^{(n_{tw}/d,n/d)}$, since $\mathcal{F}_{u_1}^{(n_{tw}/d,n/d)}\otimes\dots\otimes\mathcal{F}_{u_d}^{(n_{tw}/d,n/d)}$ is irreducible. Note that $\operatorname{ch}\left(\mathcal{F}_{u_1}^{(n_{tw}/d,n/d)}\otimes\dots\otimes\mathcal{F}_{u_d}^{(n_{tw}/d,n/d)}\right) = \prod_{k=1}^{\infty}\frac{1}{\left(1-q^{\frac{kd}{n}}\right)^d}$. Corollary 1.7.18 implies that p is injective and $\operatorname{ch}\mathcal{V}_{\lambda_1,\dots,\lambda_d}^{\mathcal{W}_q(\mathfrak{gl}_n,n_{tw})} = \prod_{k=1}^{\infty}\frac{1}{\left(1-q^{\frac{kd}{n}}\right)^d}$. □

Remark 1.7.6. Note that representation $\mathcal{F}_{u_1,\dots,u_d}^{\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})}$ is irreducible iff $\mathcal{F}_{u_1}\otimes\dots\otimes\mathcal{F}_{u_d}$ is irreducible. In particular, for $d=1$ then $\mathcal{F}_{u}^{\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})}$ is irreducible automatically. Generally, criterion of irreducibility of $\mathcal{F}_{u_1}\otimes\dots\otimes\mathcal{F}_{u_d}$ is given by Lemma 1.9.1.

Definition 1.7.5. *Verma module $\mathcal{V}_{\lambda_1,\dots,\lambda_d}^{\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})}$ is a module over $\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})$ with cyclic vector $|\bar{\lambda}\rangle_{\mathfrak{sl}}$ and relations*

$$T_k^{tw}[r]\,|\bar{\lambda}\rangle_{\mathfrak{sl}} = 0 \qquad \text{for } r>0 \tag{1.7.72}$$

$$T_{ns/d}^{tw}[0]\,|\bar{\lambda}\rangle_{\mathfrak{sl}} = \lambda_s\,|\bar{\lambda}\rangle_{\mathfrak{sl}}. \tag{1.7.73}$$

Introduce grading on $\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})$ by $\deg T_k^{tw}[r]=-r$. Verma module $\mathcal{V}_{\lambda_1,\dots,\lambda_d}^{\mathcal{W}_q(\mathfrak{gl}_n,n_{tw})}$ is a graded module with grading defined by $\deg|\bar{\lambda}\rangle_{\mathfrak{sl}}=0$.

To simplify notation for comparison of $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ cases, we will assume below that $\mu=1$ (cf. Remark 1.7.4).

Proposition 1.7.20. *$\mathcal{V}_{\lambda_1,\dots,\lambda_d}^{\mathcal{W}_q(\mathfrak{gl}_n,n_{tw})} \cong \mathcal{V}_{\lambda_1,\dots,\lambda_d}^{\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})}\otimes F^H$ with respect to identification $\mathcal{W}_q(\mathfrak{gl}_n,n_{tw}) = \mathcal{W}_q(\mathfrak{sl}_n,n_{tw})\otimes U(\mathfrak{Heis})$.*

Proof. The existence of maps in both directions can be checked directly using universal property of the Verma module. Evidently, these maps are mutually inverse. □

Corollary 1.7.21. *If $\mathcal{F}_{u_1,\dots,u_d}^{\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})}$ is irreducible then natural map $\tilde{p}\colon \mathcal{V}_{\lambda_1,\dots,\lambda_d}^{\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})} \to \mathcal{F}_{u_1,\dots,u_d}^{\mathcal{W}_q(\mathfrak{sl}_n,n_{tw})}$ is an isomorphism for λ_s as in* (1.7.66).

1.8 Restriction on $\mathfrak{Diff}_q^\Lambda$ for general sublattice

We generalize results of Section 1.5 for arbitrary sublattice. For applications in Section 1.9 we will need only case of sublattice $\Lambda_0 = \mathrm{span}(e_1, ne_2) \subset \mathbb{Z}^2$.

1.8.1 Decomposition of restriction

Let $\Lambda \subset \mathbb{Z}^2$ be a sublattice of finite index. Let us choose basis w_1, w_2 of lattice Λ so that $w_1 = (r, n_{tw})$ and $w_2 = (0, n)$. Let d be the greatest common divisor of n and n_{tw}.

Theorem 1.8.1. *There is an isomorphism of $\mathfrak{Diff}_q$-modules*

$$\mathcal{F}^{[1/nr]}_{u^{d/nr}}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)} \cong \bigoplus_{l\in\mathbf{Q}_{(d)}} \mathcal{F}^{(n_{tw}/d,\, n/d)}_{uq^{rl_0}} \otimes \dots \otimes \mathcal{F}^{(n_{tw}/d,\, n/d)}_{uq^{r(\frac{\alpha}{d}+l_\alpha)}} \otimes \dots \otimes \mathcal{F}^{(n_{tw}/d,\, n/d)}_{uq^{r(\frac{d-1}{d}+l_{d-1})}} \tag{1.8.1}$$

Proof. Proposition 1.3.3 implies that $\mathcal{F}^{[1/nr]}_{u^{d/nr}}\Big|_{\phi_{re_1,e_2}} \cong \mathcal{F}^{[1/n]}_{u^{d/n}}$. Hence it is enough to consider case $r=1$. We will use realization of $\mathcal{F}^{[1/n]}_{u^{d/n}}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)}$ constructed in Proposition 1.5.5. Strategy of our proof is as follows; first we will construct decomposition on the level of vector spaces and then study action on each direct summand.

For each $\alpha = 0, \dots, d-1$ let $\mathbf{Q}^{\alpha, l_\alpha}_{(n/d)}$ be a subset of lattice $\mathbf{P}_{(n)}$ consisting of elements

$$\sum_{a\equiv\alpha \bmod d} \tilde{l}_a Q_a \quad \text{such that} \quad \sum_{a\equiv\alpha \bmod d} \tilde{l}_a = l_\alpha.$$

Note that

$$\mathbf{Q}_{(n)} = \coprod_{l_0+\dots+l_{d-1}=0} \mathbf{Q}^{0,\,l_0}_{(n/d)} \oplus \dots \oplus \mathbf{Q}^{d-1,\,l_{d-1}}_{(n/d)}.$$

Or equivalently

$$\mathbb{C}[\mathbf{Q}_{(n)}] = \bigoplus_{l_0+\dots+l_{d-1}=0} \mathbb{C}\Big[\mathbf{Q}^{0,\,l_0}_{(n/d)}\Big] \otimes \dots \otimes \mathbb{C}\Big[\mathbf{Q}^{d-1,\,l_{d-1}}_{(n/d)}\Big].$$

Let $F^{\frac{n}{d}a,\alpha}$ be Fock module for Heisenberg algebra generated by $a_b[k]$ for $b \equiv \alpha \bmod d$. Then

$$F^{na} \otimes \mathbb{C}[\mathbf{Q}_{(n)}] = \bigoplus_{l_0+\dots+l_{d-1}=0} \Big(F^{\frac{n}{d}a,0} \otimes \mathbb{C}\Big[\mathbf{Q}^{0,\,l_0}_{(n/d)}\Big]\Big) \otimes \dots \otimes \Big(F^{\frac{n}{d}a,d-1} \otimes \mathbb{C}\Big[\mathbf{Q}^{d-1,\,l_{d-1}}_{(n/d)}\Big]\Big) \tag{1.8.2}$$

Let us show that (1.8.2) is a decomposition of $\mathfrak{Diff}_q$-modules (moreover, that it leads to decomposition (1.8.1)). Let us define

$$H^{tw}_\alpha[k] = \sum_{b\equiv\alpha \bmod d} a_b[k] \tag{1.8.3}$$

$$E^{tw}_\alpha(z) = \sum_{\substack{b-a\equiv -n_{tw} \bmod n\\ a\equiv\alpha \bmod d}} u^{\frac{d}{n}} q^{\frac{a+b-n}{2n}} z^{\frac{n_{tw}-a+b}{n}+1} :\exp\Big(\phi_b(q^{1/2}z) - \phi_a(q^{-1/2}z)\Big): \epsilon_{a,b} \tag{1.8.4}$$

$$F^{tw}_\alpha(z) = \sum_{\substack{b-a\equiv n_{tw} \bmod n\\ a\equiv\alpha \bmod d}} u^{-\frac{d}{n}} q^{\frac{-a-b+n}{2n}} z^{\frac{-n_{tw}-a+b}{n}+1} :\exp\Big(\phi_b(q^{-1/2}z) - \phi_a(q^{1/2}z)\Big): \epsilon_{a,b}, \tag{1.8.5}$$

here $\epsilon_{a,b} = \prod_r (-1)^{a_r[0]}$ (product over such r that $a-1 \geqslant r \geqslant b$ for $a > b$ and $b-1 \geqslant r \geqslant a$ for $b > a$).

Lemma 1.8.1. *Formulas* (1.8.3)–(1.8.5) *defines an action of $\mathfrak{Diff}_q$ on $F^{\frac{n}{d}a,\alpha} \otimes \mathbb{C}[\mathbf{Q}^{\alpha,\,l_\alpha}_{(n/d)}]$; obtained representation is isomorphic to $\mathcal{F}^{(n_{tw}/d,\, n/d)}_{uq^{\frac{\alpha}{d}+l_\alpha}}$.*

Sketch of the proof. Let us define $\tilde{\epsilon}_{a,b} = \prod_r (-1)^{a_r[0]}$ (product over r satisfying above inequalities and condition $r \equiv \alpha \mod d$. One can check that there exists an index set I such that conjugation of $E_\alpha^{tw}(z)$ and $F_\alpha^{tw}(z)$ by $\prod_{(i,j)\in I}(-1)^{a_i[0]a_j[0]}$ will turn $\epsilon_{a,b}$ to $\tilde{\epsilon}_{a,b}$. Theorem 1.4.2 finishes the proof. $\square$

On the other hand, formulas (1.5.27)–(1.5.29) implies

$$H^{tw}[k] = \sum_\alpha H_\alpha^{tw}[k], \quad E^{tw}(z) = \sum_\alpha E_\alpha^{tw}(z), \quad F^{tw}(z) = \sum_\alpha F_\alpha^{tw}(z). \tag{1.8.6}$$

Therefore embedding of vector space from first row of following commutative diagram leads to second row embedding of $\mathfrak{Diff}_q$-modules

$$\begin{array}{ccc}
\left(F^{\frac{n}{d}a,\,0} \otimes \mathbb{C}\left[\mathbf{Q}_{(n/d)}^{0,\,l_0}\right]\right) \otimes \cdots \otimes \left(F^{\frac{n}{d}a,\,d-1} \otimes \mathbb{C}\left[\mathbf{Q}_{(n/d)}^{d-1,\,l_{d-1}}\right]\right) & \hookrightarrow & F^{na} \otimes \mathbb{C}[\mathbf{Q}_{(n)}] \\
\| & & \| \\
\mathcal{F}_{uq^{l_0}}^{(n_{tw}/d,n/d)} \otimes \cdots \otimes \mathcal{F}_{uq^{\frac{\alpha}{d}+l_\alpha}}^{(n_{tw}/d,n/d)} \otimes \cdots \otimes \mathcal{F}_{uq^{\frac{d-1}{d}+l_{d-1}}}^{(n_{tw}/d,n/d)} & \hookrightarrow & \mathcal{F}_{u^{d/n}}^{[1/n]}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)}
\end{array}$$

$\square$

Corollary 1.8.2. *Following $\mathfrak{Diff}_q$-modules are isomorphic*

$$\mathcal{F}_u^{[1/n]}\Big|_{\phi_{e_1,ne_2}(\mathfrak{Diff}_q)} \cong \bigoplus_{l\in\mathbf{Q}_{(n)}} \mathcal{F}_{uq^{l_0}} \otimes \cdots \otimes \mathcal{F}_{uq^{\frac{\alpha}{n}+l_\alpha}} \otimes \cdots \otimes \mathcal{F}_{uq^{\frac{n-1}{n}+l_{n-1}}}. \tag{1.8.7}$$

Remark 1.8.1. Lattice Λ admits another basis $v_1 = (N,0)$, $v_2 = (R,d)$. There is an isomorphism

$$\mathcal{F}_{u^{d/nr}}^{[1/nr]}\Big|_{\phi_{v_1,v_2}(\mathfrak{Diff}_q)} \cong \bigoplus_{l\in\mathbf{Q}_{(d)}} \mathcal{F}_{uq^{rl_0}} \otimes \cdots \otimes \mathcal{F}_{uq^{r(\frac{\alpha}{n}+l_\alpha)}} \otimes \cdots \otimes \mathcal{F}_{uq^{r\left(\frac{d-1}{d}+l_{d-1}\right)}}. \tag{1.8.8}$$

1.8.2 Strange Bosonization and Odd Bosonization

Representation $\mathcal{F}_u^{[1/n]}\Big|_{\phi_{w_1,w_2}\mathfrak{Diff}_q}$ admits fermionic, bosonic, and strange bosonic realizations; formulas are given in Proposition 1.5.4, 1.5.5 and 1.5.8 correspondingly. This Section is devoted to study of corresponding $\mathcal{W}_q(\mathfrak{sl}_n, n_{tw})$ algebra action on $\mathcal{F}_u^{[1/n]}\Big|_{\phi_{w_1,w_2}\mathfrak{Diff}_q}$. We will consider strange bosonization and its classical limit.

Let us introduce following notation (cf. with (1.3.2))

$$\mathsf{e}(z) = :\exp\left(\sum_{k\neq 0} \frac{q^{-k/2n} - q^{k/2n}}{k} a_k z^{-k}\right): \tag{1.8.9}$$

Proposition 1.8.3. *For $w_1 = e_1 + n_{tw}e_2$, $w_2 = ne_2$ ideal $J_{\mu,n,n_{tw}}$ acts by zero on $\mathcal{F}_u^{[1/n]}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)}$ for*

$$\mu = \frac{u}{n(1-q^{\frac{1}{n}})} \times (-1)^{\frac{d}{n}} \frac{q^{-\frac{1}{2n}} - q^{\frac{1}{2n}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} n = (-1)^{\frac{d}{n}} \frac{q^{-\frac{1}{2n}}u}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} \tag{1.8.10}$$

Proof. We will use strange bosonization of $\mathcal{F}_u^{[1/n]}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)}$. Recall that we denote by ζ a primitive root of unity of degree n. We have

$$E(z) \mapsto \frac{u}{n(1-q^{\frac{1}{n}})} z^{\frac{n_{tw}}{n}} \left(\mathsf{e}(z^{1/n}) + \zeta^{n_{tw}}\mathsf{e}(\zeta z^{1/n}) + \cdots + \zeta^{(n-1)n_{tw}}\mathsf{e}(\zeta^{n-1}z^{1/n})\right) \tag{1.8.11}$$

Let us calculate

$$\left(\frac{n(1-q^{\frac{1}{n}})}{u}E(z)\right)^n = z^{n_{tw}}\left(\mathsf{e}(z^{1/n}) + \zeta^{n_{tw}}\mathsf{e}(\zeta z^{1/n}) + \dots + \zeta^{(n-1)n_{tw}}\mathsf{e}(\zeta^{n-1}z^{1/n})\right)^n =$$

$$n! z^{n_{tw}}(-1)^{(n-1)n_{tw}}\mathsf{e}(z^{1/n})\mathsf{e}(\zeta z^{1/n})\cdots\mathsf{e}(\zeta^{n-1}z^{1/n}) =$$

$$(-1)^{(n-1)n_{tw}} n! z^{n_{tw}} \prod_{i<j}\frac{(1-\zeta^{j-i})^2}{((1-q^{\frac{1}{n}}\zeta^{j-i})(1-q^{-\frac{1}{n}}\zeta^{j-i}))} :\mathsf{e}(z^{1/n})\mathsf{e}(\zeta z^{1/n})\cdots\mathsf{e}(\zeta^{n-1}z^{1/n}):$$

We need to compute

$$\prod_{i<j}\frac{(1-\zeta^{j-i})^2}{((1-q^{\frac{1}{n}}\zeta^{j-i})(1-q^{-\frac{1}{n}}\zeta^{j-i})} = \prod_{i<j}\frac{(1-\zeta^{j-i})(1-\zeta^{i-j})}{q^{-\frac{1}{n}}(1-q^{\frac{1}{n}}\zeta^{j-i})(1-q^{\frac{1}{n}}\zeta^{i-j})} = q^{\frac{1}{n}\binom{n}{2}}\prod_{k\neq 0}\frac{(1-\zeta^k)^n}{((1-q^{\frac{1}{n}}\zeta^k)^n} =$$

$$q^{\frac{n-1}{2}}\frac{(1-q^{\frac{1}{n}})^n}{(1-q)^n}n^n = \frac{(q^{\frac{1}{2n}}-q^{-\frac{1}{2n}})^n}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^n}n^n$$

So

$$\left(\frac{n(1-q^{\frac{1}{n}})}{u}E(z)\right)^n = n! z^{n_{tw}}(-1)^{(n-1)n_{tw}}\left(\frac{(q^{\frac{1}{2n}}-q^{-\frac{1}{2n}})}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})}n\right)^n :\mathsf{e}(z^{1/n})\mathsf{e}(\zeta z^{1/n})\cdots\mathsf{e}(\zeta^{n-1}z^{1/n}): \tag{1.8.12}$$

Note that $:\mathsf{e}(z^{1/n})\mathsf{e}(\zeta z^{1/n})\cdots\mathsf{e}(\zeta^{n-1}z^{1/n}):=:\exp\varphi(z):$. Hence

$$\mu^n = (-1)^{(n-1)n_{tw}}\left(\frac{u}{n(1-q^{\frac{1}{n}})}\frac{(q^{\frac{1}{2n}}-q^{-\frac{1}{2n}})}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})}n\right)^n = (-1)^{(n-1)n_{tw}+n}\left(\frac{uq^{-\frac{1}{2n}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\right)^n \tag{1.8.13}$$

Finally note that $(n-1)n_{tw}+n \equiv d \mod 2$. □

Remark 1.8.2. Another way to prove Proposition 1.8.3 is to derive it from Proposition 1.7.12 (since isomorphism (1.8.1)). Beware inconsistency of our notation in (1.7.48) and (1.8.10). Let us rewrite (1.7.48)

$$\mu = (-1)^{\frac{d}{n}}\frac{q^{-\frac{d}{2n}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}(u_1\dots u_d)^{\frac{1}{n}} = (-1)^{\frac{d}{n}}\frac{q^{-\frac{d}{2n}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\left(\left(u^{\frac{n}{d}}\right)^d q^{\frac{d-1}{2}}\right)^{\frac{1}{n}} = (-1)^{\frac{d}{n}}\frac{q^{-\frac{1}{2n}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}u$$

Let us consider subalgebra of Heisenberg algebra generated by $J_k = a_k$ for $n\nmid k$. Denote corresponding Fock module by F^J.

Corollary 1.8.4. *There is an action of $\mathcal{W}_q(\mathfrak{sl}_n, n_{tw})$ on F^J given by*

$$T_1(z) = -(-1)^{\frac{d}{n}}\frac{1}{n}\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{q^{\frac{1}{2n}}-q^{-\frac{1}{2n}}}z^{\frac{n_{tw}}{n}}\sum_{l=0}^{n-1}\zeta^{ln_{tw}} :\exp\left(\sum_{n\nmid k}\frac{q^{-\frac{k}{2n}}-q^{\frac{k}{2n}}}{k}J_k\zeta^{-kl}z^{-\frac{k}{n}}\right): \tag{1.8.14}$$

Obtained representation corresponds to $\mathcal{F}_u^{[1/n]}\Big|_{\phi_{w_1,w_2}(\mathfrak{Diff}_q)}$.

Proof. Follows from Theorem 1.7.1 and Proposition 1.8.3 □

Remark 1.8.3. Let us consider non-twisted case $n_{tw}=0$. Then $d=n$ and total sign $-(-1)^{\frac{d}{n}}=1$. More accurately, the coefficient is a root of unity of degree n (cf. Remark 1.7.4). Nevertheless, this freedom disappears if we require $T_1(z) = n + o(\hbar)$ for $\hbar = \log q$ (this is a standard setting for classical limit).

Example 1.8.1. Odd bosonization is a particular case of strange bosonization for $n = 2$ and $n_{tw} = 0$.

$$T_1(z) = \frac{q^{\frac14}+q^{-\frac14}}{2}\left[:\exp\left(\sum_{2\nmid r}\frac{q^{-\frac{r}{4}}-q^{\frac{r}{4}}}{r}J_r z^{-\frac{r}{2}}\right): + :\exp\left(-\sum_{2\nmid r}\frac{q^{-\frac{r}{4}}-q^{\frac{r}{4}}}{r}J_r z^{-\frac{r}{2}}\right):\right] \tag{1.8.15}$$

Consider classical limit $q \to 1$. It is convenient to assume $q = e^{\hbar}$ and $\hbar \to 0$. If there exists an expansion

$$T_1(z) = 2 + z^2 L(z)\hbar^2 + o(\hbar^2) \tag{1.8.16}$$

then modes of current $L(z) = L_n z^{-n-2}$ form 'not q-deformed' Virasoro algebra. Note that

$$T_1(z) \to 2 + z^2\left(\frac{z^{-2}}{16} + \frac14 \sum :J_{\text{odd}}(z)^2:\right)\hbar^2 + o(\hbar^2) \tag{1.8.17}$$

where $J_{\text{odd}}(z) = \sum_{2\nmid r} J_r z^{-\frac{r}{2}-1}$. Hence

$$L(z) = \frac{z^{-2}}{16} + \frac14 \sum :J_{\text{odd}}(z)^2: . \tag{1.8.18}$$

Or equivalently

$$L_k = \frac14 \sum_{\frac12(r+s)=k} :J_r J_s: + \frac{1}{16}\delta_{k,0}. \tag{1.8.19}$$

Formula (1.8.19) is well-known; it coincides with [Zam87, eq.(2.16)] after substitution $J_r = 2I_{\frac12 r}$. Let us emphasize that formula (1.8.15) is a q-deformation of (1.8.19).

1.9 Relations on conformal blocks

1.9.1 Whittaker vector

In this section we define and study basic properties of Whittaker vector $W(z|u_1,\dots,u_N) \in \mathcal{F}_{u_1}\otimes\dots\otimes\mathcal{F}_{u_N}$. We will restrict ourself to case when $\mathcal{F}_{u_1}\otimes\dots\otimes\mathcal{F}_{u_N}$ is irreducible.

Lemma 1.9.1. *$\mathcal{F}_{u_1}\otimes\dots\otimes\mathcal{F}_{u_N}$ is irreducible if $u_i/u_j \neq q^k$ for any $k \in \mathbb{Z}$.*

Proof. Follows from the proof of [FFJ+11b, Lemma 3.1]. □

In papers [Neg15a] [Tsy17] Whittaker vector is defined geometrically. We will define Whittaker vector by algebraic properties (cf. [Neg15a, Prop. 4.15]). Then we will prove that these properties define Whittaker vector uniquely up to normalization if the module $\mathcal{F}_{u_1}\otimes\dots\otimes\mathcal{F}_{u_N}$ is irreducible.

Definition 1.9.1. *Whittaker vector $W(z|u_1,\dots,u_N) \in \mathcal{F}_{u_1}\otimes\dots\otimes\mathcal{F}_{u_N}$ is an eigenvector of operators $E_{a,b}$ for $Nb \geqslant a \geqslant 0$ with eigenvalues*

$$E_{0,k}\, W(z|u_1,\dots,u_N) = \frac{z^k}{q^{k/2}-q^{-k/2}} W(z|u_1,\dots,u_N) \tag{1.9.1}$$

$$E_{Nk,k}\, W(z|u_1,\dots,u_N) = \frac{\left((-q^{-\frac12})^N u_1\cdots u_N z\right)^k}{q^{-k/2}-q^{k/2}} W(z|u_1,\dots,u_N) \tag{1.9.2}$$

for $k > 0$;

$$E_{k_1,k_2} W(z|u_1,\dots,u_N) = 0 \tag{1.9.3}$$

for $Nk_2 > k_1 > 0$. We require $W(z|u_1,\dots,u_N) = |\varnothing\rangle\otimes\dots\otimes|\varnothing\rangle + \cdots$ to fix normalization (by dots we mean lower vectors).

Remark 1.9.1. Whittaker vector is an element of graded completion of $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_N}$. Abusing notation, we use the same symbols for modules and their completions.

Remark 1.9.2. Whittaker vector is an eigenvector for surprisingly big algebra. This explains why we have to consider specific eigenvalues (for general eigenvalues there is no eigenvector in corresponding representation). Theorem 1.13.2 clarify origin of this eigenvalues.

Theorem 1.9.1. *If $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_N}$ is irreducible, then there exists unique Whittaker vector.*

One can find a proof of Theorem 1.9.1 in Section 1.13. This statement can be considered as a part of folklore; unfortunately, we do not know a precise reference for the theorem.

Proposition 1.9.2. *Decomposition of Whittaker vector $W(z^{1/n}|u) \in \mathcal{F}_u^{[1/n]}$ with respect to (1.8.7) is given by Whittaker vectors $W(z|uq^{l_0}, \ldots, uq^{\frac{n-1}{n}+l_{n-1}})$ up to normalization.*

Proof. The idea of the proof is that relations (1.9.1) and (1.9.2) for $N = 1$ implies these relations for $N = n$. Let us work out conditions (1.9.2) for $W(z|uq^{l_0}, \ldots, uq^{\frac{n-1}{n}+l_{n-1}})$

$$E_{nk,k}\, W(z|uq^{l_0}, \ldots, uq^{\frac{n-1}{n}+l_{n-1}}) = \frac{\left(\left(-q^{-\frac{1}{2}}\right)^n \prod_k \left(uq^{\frac{k}{n}+l_k}\right) z\right)^k}{(q^{-k/2} - q^{k/2})} W(z|uq^{l_0}, \ldots, uq^{\frac{n-1}{n}+l_{n-1}}) \tag{1.9.4}$$

Let us calculate

$$\left(-q^{-\frac{1}{2}}\right)^n \prod_{k=0}^{n-1} \left(uq^{\frac{k}{n}+l_k}\right) = \left(-q^{-\frac{1}{2}}\right)^n u^n q^{\frac{n-1}{2}} = \left(-q^{-\frac{1}{2n}} u\right)^n \tag{1.9.5}$$

So $W(z|uq^{l_0}, \ldots, uq^{\frac{n-1}{n}+l_{n-1}})$ is defined (up to normalization) by conditions

$$E_{0,k}\, W(z|uq^{l_0}, \ldots, uq^{\frac{n-1}{n}+l_{n-1}}) = \frac{z^k}{q^{k/2} - q^{-k/2}} W(z|uq^{l_0}, \ldots, uq^{\frac{n-1}{n}+l_{n-1}}) \tag{1.9.6}$$

$$E_{nk,k}\, W(z|uq^{l_0}, \ldots, uq^{\frac{n-1}{n}+l_{n-1}}) = \frac{\left(-q^{-\frac{1}{2n}} z^{\frac{1}{n}} u\right)^{nk}}{q^{-k/2} - q^{k/2}} W(z|uq^{l_0}, \ldots, uq^{\frac{n-1}{n}+l_{n-1}}) \tag{1.9.7}$$

$$E_{k_1,k_2} W(z|uq^{l_0}, \ldots, uq^{\frac{n-1}{n}+l_{n-1}}) = 0 \tag{1.9.8}$$

for $k > 0$ and $nk_2 > k_1 > 0$. Denote by $E_{a,b}^{[1/n]}$ generators of $\mathfrak{Diff}_{q^{1/n}}$. Then

$$E_{0,k} W(z^{1/n}|u) = E_{0,nk}^{[1/n]} W(z^{1/n}|u) = \frac{\left(z^{1/n}\right)^{nk}}{\left(q^{1/n}\right)^{nk/2} - \left(q^{1/n}\right)^{-nk/2}} W(z^{1/n}|u) \tag{1.9.9}$$

$$E_{nk,k} W(z^{1/n}|u) = E_{nk,nk}^{[1/n]} W(z^{1/n}|u) = \frac{\left(-q^{-\frac{1}{2n}} z^{\frac{1}{n}} u\right)^{nk}}{\left(q^{1/n}\right)^{-nk/2} - \left(q^{1/n}\right)^{nk/2}} W(z^{1/n}|u) \tag{1.9.10}$$

$$E_{k_1,k_2} W(z^{1/n}|u) = E_{k_1,nk_2}^{[1/n]} W(z^{1/n}|u) = 0 \tag{1.9.11}$$

Note that conditions (1.9.9)–(1.9.11) and conditions (1.9.6)–(1.9.8) coincide. Hence each component of $W(z^{1/n}|u)$ also satisfy those conditions, i.e. coincide with Whittaker vector up to normalization. □

Whittaker vector for $\mathcal{F}_u$

Recall that we use notation $c(\lambda) = \sum_{s\in\lambda} c(s)$.

Proposition 1.9.3. *We have an expansion of vector $W(z|u)$ in the basis $|\lambda\rangle$*

$$W(z|u) = \sum \frac{q^{-\frac{1}{2}c(\lambda)}}{\prod_{s\in\lambda}\left(q^{\frac{1}{2}h(s)} - q^{-\frac{1}{2}h(s)}\right)} z^{|\lambda|}|\lambda\rangle \tag{1.9.12}$$

To prove the Proposition we need the following lemmas.

Lemma 1.9.4. *Following vectors in $\mathcal{F}_u$ coincide*

$$\exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k(q^{k/2} - q^{-k/2})} a_{-k}\right)|\varnothing\rangle = \sum \frac{q^{-\frac{1}{2}c(\lambda)}}{\prod_{s\in\lambda}\left(q^{\frac{1}{2}h(s)} - q^{-\frac{1}{2}h(s)}\right)} z^{|\lambda|}|\lambda\rangle \tag{1.9.13}$$

Proof. Recall Cauchy identity

$$\exp\left(-\sum_k \frac{1}{k} p_k(x)p_k(y)\right) = \prod_{i,j}(1 - x_i y_j) = \sum_\lambda (-1)^{|\lambda|} s_{\lambda'}(x)s_\lambda(y). \tag{1.9.14}$$

Let us use specialization of Cauchy identity (see [Mac95, §1.4 Ex. 2])

$$p_k(x) \mapsto \frac{-z^k}{q^{k/2} - q^{-k/2}}, \qquad s_{\lambda'}(x) \mapsto (-1)^{|\lambda|} \frac{q^{-\frac{1}{2}c(\lambda)} z^{|\lambda|}}{\prod_{s\in\lambda}\left(q^{\frac{1}{2}h(s)} - q^{-\frac{1}{2}h(s)}\right)}. \tag{1.9.15}$$

To finish the proof let us recall that there is an identification of space of symmetric polynomials and Fock module F^a_α given by $s_\lambda \mapsto |\lambda\rangle$ and $p_k \mapsto a_{-k}$ (see [KR87]). □

Remark 1.9.3. For $|q| < 1$ this specialization comes from substitution $p_k(zq^{1/2}, zq^{3/2}, \ldots)$.

Lemma 1.9.5. *Following vectors in $\mathcal{F}_u$ coincide*

$$\exp\left(-\sum_{k=1}^{\infty} \frac{\left(-q^{-1/2}uz\right)^k}{k(q^{k/2} - q^{-k/2})} \rho(E_{-k,-k})\right)|\varnothing\rangle = \sum \frac{q^{-\frac{1}{2}c(\lambda)}}{\prod_{s\in\lambda}\left(q^{\frac{1}{2}h(s)} - q^{-\frac{1}{2}h(s)}\right)} z^{|\lambda|}|\lambda\rangle \tag{1.9.16}$$

Proof. Recall that we have defined an operator I_τ by (1.3.23). Let us calculate

$$\mathrm{I}_\tau\left(\sum \frac{q^{-\frac{1}{2}c(\lambda)}}{\prod_{s\in\lambda}\left(q^{\frac{1}{2}h(s)} - q^{-\frac{1}{2}h(s)}\right)} z^{|\lambda|}|\lambda\rangle\right) = \sum \frac{u^{|\lambda|} q^{\frac{1}{2}c(\lambda) - \frac{1}{2}|\lambda|}}{\prod_{s\in\lambda}\left(q^{\frac{1}{2}h(s)} - q^{-\frac{1}{2}h(s)}\right)} z^{|\lambda|}|\lambda\rangle \tag{1.9.17}$$

Proposition 1.3.8 implies that

$$\mathrm{I}_\tau\left(\exp\left(-\sum_{k=1}^{\infty} \frac{\left(-q^{-1/2}uz\right)^k}{k(q^{k/2} - q^{-k/2})} \rho(E_{-k,-k})\right)|\varnothing\rangle\right) = \exp\left(-\sum_{k=1}^{\infty} \frac{\left(-q^{-1/2}uz\right)^k}{k(q^{k/2} - q^{-k/2})} a_{-k}\right)|\varnothing\rangle \tag{1.9.18}$$

Using Cauchy identity for another specialization

$$p_k(x) \mapsto \frac{\left(-q^{-1/2}uz\right)^k}{q^{k/2} - q^{-k/2}}, \qquad s_{\lambda'}(x) \mapsto (-1)^{|\lambda|} \frac{u^{|\lambda|} q^{\frac{1}{2}c(\lambda) - \frac{1}{2}|\lambda|}}{\prod_{s\in\lambda}\left(q^{\frac{1}{2}h(s)} - q^{-\frac{1}{2}h(s)}\right)} z^{|\lambda|}, \tag{1.9.19}$$

we see that RHS of (1.9.17) and (1.9.18) coincide. □

Remark 1.9.4. For $|q| > 1$ this specialization comes from substitution $p_k(-zuq^{-1}, -zuq^{-2}, \ldots)$.

Proof of Proposition 1.9.3. To prove the Proposition let us check that RHS of (1.9.12) satisfy condition (1.9.1)–(1.9.3). Conditions (1.9.1) and (1.9.2) are equivalent to Lemmas 1.9.4 and 1.9.5 correspondingly. To finish the proof we note that for $N = 1$ conditions (1.9.1) and (1.9.2) imply (1.9.3). □

Remark 1.9.5. Note that we did not use Theorem 1.9.1 in the proof of Proposition 1.9.3. Moreover, we have proven a particular case of the Theorem for $W(z|u)$.

Whittaker vector and restriction on sublattice

Let us recall interpretation of decomposition (1.8.7) in terms of boson-fermion correspondence. One can identify $F^{n\psi} = F^{\psi}$. Embedding $F_0^a \subset F^{\psi}$ corresponded to embedding $F^{na} \otimes \mathbb{C}[\mathbf{Q}_{(n)}] \subset F^{n\psi}$. Hence we have decomposition

$$F^a = \bigoplus_{l \in \mathbf{Q}_{(n)}} F^{na} \otimes e^{\sum_i l_i Q_i} \tag{1.9.20}$$

We argue by construction that decomposition (1.8.7) correspond to (1.9.20).

Proposition 1.9.6. *Decomposition* (1.9.20) *identifies* $|\varnothing\rangle \otimes e^{-\sum_i l_i Q_i}$ *with* $|\lambda\rangle$ *for some* λ. *Moreover, partition* λ *satisfy following properties.*

(i) Hooks of λ *are in bijection with tuples* $\{(i, j, k_i, k_j) \mid k_i < l_i;\ k_j \geqslant l_j;\ nk_i + i > nk_j + j\}$. *Length of a hook corresponding to a tuple* (i, j, k_i, k_j) *equals to* $n(k_i - k_j) + i - j$

(ii) $\frac{1}{2}\sum_{i=0}^{n-1}(\frac{i}{n} + l_i)^2 = \frac{|\lambda|}{n} + \frac{1}{2}\sum_{i=0}^{n-1}\frac{i^2}{n^2}$.

Proof. The n-fermion Fock space $F^{n\psi}$ is isomorphic to tensor product $F^{\psi} \otimes \cdots \otimes F^{\psi}$. The n-Heisenberg highest vectors are products $|l_0\rangle \otimes \cdots \otimes |l_{n-1}\rangle$. After identification of $F^{n\psi}$ with one F^{ψ}, these products becomes (1.3.16) for special λ. Such diagrams λ are called n-cores.

Combinatorially boson-fermion correspondence is a correspondence between Maya diagrams and charged partitions (λ, l), see e.g. [Neg15b, Section 6.4] or [FM17]. Boxes of a partition correspond to pairs of white and black points in Maya diagram such that the coordinate of white point is greater than the coordinate of the black point. The hook length equals the difference between the coordinates of white and black points (cf. [Neg15b, Section 6.4]). This proves (i).

For formula (ii) see e.g. [FM17, Prop. 2.30]. □

Lemma 1.9.7. *Let* $l_i > l_j$. *Let us consider hooks with fixed* i *and* j *(see Proposition 1.9.6).*

- *If* $i > j$, *then possible lengths of hooks are* $nk + i - j$ *for* $k = 0, 1, \ldots, l_i - l_j - 1$.
- *If* $i < j$ *then possible lengths of hooks are* $nk + i - j$ *for* $k = 1, 2, \ldots, l_i - l_j - 1$.

There are exactly $l_i - l_j - k$ *such hooks of length* $nk + i - j$ *for all possible* k.

For each $l \in \mathbf{Q}_{(n)}$ we will use notation $\prod_{(i,j,k)}$ for product over triples (i, j, k) satisfying following conditions. Numbers i, j run over $0, \ldots, n-1$ with condition $l_i > l_j$. If $i > j$, then $k = 0, 1, \ldots, l_i - l_j - 1$; if $i < j$ then $k = 1, 2, \ldots, l_i - l_j - 1$.

Corollary 1.9.8. *If diagram* λ *corresponds to* $|\varnothing\rangle \otimes e^{-\sum_i l_i Q_i}$, *then*

$$\prod_{s \in \lambda}\left(q^{\frac{1}{2}h(s)} - q^{-\frac{1}{2}h(s)}\right) = \prod_{(i,j,k)}\left(q^{\frac{1}{2}(nk+i-j)} - q^{-\frac{1}{2}(nk+i-j)}\right)^{l_i - l_j - k}. \tag{1.9.21}$$

Theorem 1.9.2. *Decomposition of Whittaker vector* $W(z^{1/n}|u^{1/n}) \in \mathcal{F}^{[1/n]}_{u^{1/n}}$ *with respect to* (1.8.7) *is given by*

$$\frac{q^{-\frac{1}{2n}c(\lambda)} z^{\frac{1}{2}\sum\left((l_i+\frac{i}{n})^2-(\frac{i}{n})^2\right)}}{\prod_{(i,j,k)}\left(q^{\frac{1}{2}(k+\frac{i-j}{n})}-q^{-\frac{1}{2}(k+\frac{i-j}{n})}\right)^{l_i-l_j-k}} W(z|uq^{l_0},\ldots,uq^{\frac{n-1}{n}+l_{n-1}}) \tag{1.9.22}$$

Proof. Recall that according to Proposition 1.9.2 we just have to verify the coefficient. This coefficient can be found as coefficient of $W(z^{1/n}|u^{1/n})$ at highest vector of $\mathcal{F}_{uq^{l_0}}\otimes\cdots\otimes\mathcal{F}_{uq^{\frac{k}{n}+l_k}}\otimes\cdots\otimes\mathcal{F}_{uq^{\frac{n-1}{n}+l_{n-1}}}$.

By Proposition 1.9.3, coefficient of $W(z^{1/n}|u^{1/n})$ at $|\lambda\rangle$ is

$$\frac{q^{-\frac{1}{2n}c(\lambda)}\, z^{\frac{|\lambda|}{n}}}{\prod_{s\in\lambda}\left(q^{\frac{1}{2n}h(s)}-q^{-\frac{1}{2n}h(s)}\right)} = \frac{q^{-\frac{1}{2n}c(\lambda)} z^{\frac{1}{2}\sum\left((l_i+\frac{i}{n})^2-(\frac{i}{n})^2\right)}}{\prod_{(i,j,k)}\left(q^{\frac{1}{2}(k+\frac{i-j}{n})}-q^{-\frac{1}{2}(k+\frac{i-j}{n})}\right)^{l_i-l_j-k}}. \tag{1.9.23}$$

Equality (1.9.23) follows from Corollary 1.9.8 and Proposition 1.9.6 (ii). □

1.9.2 Shapovalov form

Definition 1.9.2. *Let* M_1*,* M_2 *be two representations of* $\mathfrak{Diff}_q$*. A pairing* $\langle -,-\rangle_s : M_1\otimes M_2 \to \mathbb{C}$ *is called Shapovalov if* $\langle v, E_{a,b}w\rangle_s = -\langle E_{-a,-b}v, w\rangle_s$*.*

Proposition 1.9.9. *There exists a unique Shapovalov pairing* $\mathcal{F}_u\otimes\mathcal{F}_{qu^{-1}}\to\mathbb{C}$ *such that* $\langle 0|0\rangle_s = 1$*.*

Proof. There exists a unique pairing on Fock space such that a_k is dual to $-a_{-k}$. Since algebra $\mathfrak{Diff}_q$ is generated by modes of $E(z)$ and $F(z)$, it remains to check Shapovalov property for them. Formulas (1.3.2) and (1.3.3) implies $\langle v, E(z)w\rangle = -\langle F(z^{-1})v, w\rangle$. □

Remark 1.9.6. Note that this pairing differs from the pairing defined in Section 1.3.1. More precisely, in Section 1.3.1 we required a_k to be dual to a_{-k}, not $-a_{-k}$.

Proposition 1.9.10. $\big\langle W(1|qu^{-1}), W(z|u)\big\rangle = (qz;q,q)_\infty$

Proof. Formulas (1.9.12) and (1.9.13) implies

$$W(z|u) = \exp\left(\sum_{k=1}^{\infty}\frac{z^k}{k(q^{k/2}-q^{-k/2})}a_{-k}\right)|\varnothing\rangle. \tag{1.9.24}$$

Using (1.9.24) one can finish the by straightforward computation. □

Proposition 1.9.11. *There exists a unique Shapovalov pairing* $\mathcal{M}_u\otimes\mathcal{M}_{u^{-1}}\to\mathbb{C}$ *such that*

$$\langle 1|0\rangle_s = 1 \quad \langle v, \psi_i w\rangle = \langle\psi_{-i}v, w\rangle_s \quad \langle v, \psi_i^* w\rangle = \langle\psi_{-i}^* v, w\rangle_s \tag{1.9.25}$$

Proof. There exists unique pairing satisfying (1.9.25). The Shapovalov property can be checked directly using (1.3.12)–(1.3.14). □

Proposition 1.9.12. *Shapovalov pairing for Fock modules for basis* $|\lambda\rangle$ *has form* $\langle\lambda|\mu\rangle_s = (-1)^{|\lambda|}\delta_{\lambda,\mu'}$*.*

Proof. Let $p_1,\ldots,p_i$ and $q_1,\ldots,q_i$ be Frobenius coordinates of λ; analogously, $\tilde p_1,\ldots,\tilde p_j$ and $\tilde q_1,\ldots,\tilde q_j$ be Frobenius coordinates of μ. Using identification given by (1.3.16) we obtain

$$\langle\mu,1|\lambda,0\rangle_s = (-1)^{\sum_k(q_k-1)}(-1)^{\sum_k(\tilde q_k-1)}\langle 1|\psi^*_{\tilde q_1}\cdots\psi^*_{\tilde q_i}\psi_{\tilde p_i-1}\cdots\psi_{\tilde p_1-1}\psi_{-p_1}\cdots\psi_{-p_i}\,\psi^*_{-q_i+1}\cdots\psi^*_{-q_1+1}|\varnothing\rangle_s \tag{1.9.26}$$

Evidently, if this product is non-zero, then $i=j$, $q_k=\tilde p_k$, and $p_l=\tilde q_l$; this exactly means that $\mu=\lambda'$. It remains to calculate RHS of (1.9.26) in this case; it equals $(-1)^{\sum_k q_k+\sum_l p_l+i^2} = (-1)^{|\lambda|}$ since $|\lambda| = \sum_k q_k+\sum_l p_l - i$. □

Definition 1.9.3. *Standard Shapovalov pairing* $\langle -,-\rangle_{ss}\colon M_1\otimes M_2\to\mathbb{C}$ *for* $M_1=\mathcal{F}_{u_1}\otimes\cdots\otimes\mathcal{F}_{u_n}$ *and* $M_2=\mathcal{F}_{q/u_n}\otimes\cdots\otimes\mathcal{F}_{q/u_1}$ *is defined by*

$$\langle x_1\otimes\cdots\otimes x_n,\, y_n\otimes\cdots\otimes y_1\rangle_{ss}=\prod_i\langle x_i,y_i\rangle_i \tag{1.9.27}$$

Here $\langle -,-\rangle_i$ *stands for Shapovalov pairing* $\mathcal{F}_{u_i}\otimes\mathcal{F}_{qu_i^{-1}}\to\mathbb{C}$ *as in Proposition 1.9.9.*

Proposition 1.9.13. *Shapovalov pairing on* $\mathcal{F}_u^{[1/n]}\otimes\mathcal{F}^{[1/n]}_{q^{\frac{1}{n}}/u}$ *restricts to* $(-1)^{|\lambda|}\langle -,-\rangle_{ss}\colon M_1\otimes M_2\to\mathbb{C}$ *for* $M_1=\mathcal{F}_{uq^{l_0}}\otimes\cdots\otimes\mathcal{F}_{uq^{\frac{n-1}{n}+l_{n-1}}}$ *and* $M_2=\mathcal{F}_{q^{\frac{1}{n}-l_{n-1}}/u}\otimes\cdots\otimes\mathcal{F}_{q^{1-l_0}/u}$ *(with respect to decomposition* (1.8.7)*). Other pairs of direct summands are orthogonal.*

Proof. Property $\langle v,E_{a,b}w\rangle_s=-\langle E_{-a,-b}v,w\rangle_s$ is preserved under restriction. Since module M_1 and M_2 are irreducible, there exists unique Shapovalov pairing $M_1\otimes M_2\to\mathbb{C}$. So restriction of pairing coincides with the standard pairing up to multiplicative constant. The constant equals to $\langle\lambda'|\lambda\rangle=(-1)^{|\lambda|}$ due to Proposition 1.9.12 (and Proposition 1.9.6).

Orthogonality with all other summands also follows from Proposition 1.9.12 and irreducibility. □

Remark 1.9.7. Let us comment on another way to prove orthogonality mentioned in Proposition 1.9.13. All direct summands are pairwise non-isomorphic. Hence there is no non-zero pairing for all other pairs of direct summands.

1.9.3 Conformal blocks

Definition 1.9.4. *Pochhammer and double Pochhammer symbols are defined by*

$$(u;q_1)_\infty=\prod_{i=0}^{\infty}(1-q_1^iu),\quad (u;q_1,q_2)_\infty=\prod_{i,j=0}^{\infty}(1-q_1^iq_2^ju). \tag{1.9.28}$$

Remark 1.9.8. Standard definition works for $|q_1|,|q_2|<1$ and any u. For sufficiently small u double Pochhammer symbol can be presented as $(u;q_1,q_2)_\infty=\exp\Big(-\sum_{k=1}^\infty\frac{u^k}{k(1-q_1^k)(1-q_2^k)}\Big)$. The series in u has non-zero radios of convergence for $|q_1|,|q_2|\neq 1$. Moreover the series enjoys property $(u;q_1^{-1},q_2)_\infty=1/(q_1u;q_1,q_2)_\infty$, hence we can define double Pochhammer symbol for any $|q_1|,|q_2|\neq1$.

In particular, new definition implies $(u;q,q^{-1})_\infty=1/(qu;q,q)_\infty$; this is important to compare our formulas with [BGM19]. Below we assume $|q|\neq1$.

Definition 1.9.5. *Let us define q-deformed conformal block*

$$\mathcal{Z}(u_1,\dots,u_n;z)=z^{\frac{\sum(\log u_i)^2}{2(\log q)^2}}\prod_{i\neq j}\frac{1}{(qu_iu_j^{-1};q,q)_\infty}\big\langle W_u(1|qu_n^{-1},\dots,qu_1^{-1}),W(z|u_1,\dots,u_n)\big\rangle_{ss} \tag{1.9.29}$$

Remark 1.9.9. AGT statement claims that function $\mathcal{Z}(u_1,\dots,u_n;z)$ is equal to Nekrasov partition function for pure supersymmetric $SU(n)$ 5d theory. This was conjectured in [AY10], the proof follows from the geometric construction of the Whittaker vector given in the [Neg15a] and [Tsy17].

Theorem 1.9.3.

$$z^{\frac{1}{2}\sum\frac{i^2}{n^2}}\prod_{i\neq j}\frac{1}{(q^{1+\frac{i-j}{n}};q,q)_\infty}\left(q^{\frac{1}{n}}z^{\frac{1}{n}};q^{\frac{1}{n}},q^{\frac{1}{n}}\right)_\infty=\sum_{(l_0,\dots,l_{n-1})\in\mathbf{Q}_{(n)}}\mathcal{Z}\left(q^{l_0},q^{\frac{1}{n}+l_1},\dots,q^{\frac{n-1}{n}+l_{n-1}};z\right) \tag{1.9.30}$$

The idea of the proof is to find two different expressions for $\langle W(1|q^{1/n}), W(z^{1/n}|1)\rangle$ using Theorem 1.9.2. To do this we need to simplify $\mathcal{Z}(u_1,\ldots,u_n;z)$ after substitution $u_{i-1} = q^{\frac{i}{n}+l_i}$. Let us concentrate on the second factor of (1.9.29).

Proposition 1.9.14.

$$\prod_{i\neq j} \frac{\left(q^{1+\frac{i-j}{n}};q,q\right)_\infty}{\left(q^{l_i-l_j+1+\frac{i-j}{n}};q,q\right)_\infty} = (-1)^{|\lambda|} \prod_{(i,j,k)} \frac{1}{\left(q^{\frac{k}{2}+\frac{i-j}{2n}} - q^{-\frac{k}{2}-\frac{i-j}{2n}}\right)^{2(l_i-l_j-k)}} \tag{1.9.31}$$

Proof. Let $l_i - l_j > 0$. It is straightforward to check that

$$\frac{\left(q^{1+\frac{i-j}{n}};q,q\right)_\infty}{\left(q^{l_i-l_j+1+\frac{i-j}{n}};q,q\right)_\infty} = \left(q^{1+\frac{i-j}{n}};q\right)_\infty^{l_i-l_j} \prod_{k=1}^{l_i-l_j-1} \frac{1}{\left(1-q^{k+\frac{i-j}{n}}\right)^{l_i-l_j-k}} \tag{1.9.32}$$

$$\frac{\left(q^{1+\frac{j-i}{n}};q,q\right)_\infty}{\left(q^{l_j-l_i+1+\frac{j-i}{n}};q,q\right)_\infty} = \left(q^{1+\frac{j-i}{n}};q\right)_\infty^{l_j-l_i} \prod_{k=0}^{l_i-l_j-1} \frac{1}{\left(1-q^{-k-\frac{i-j}{n}}\right)^{l_i-l_j-k}} \tag{1.9.33}$$

Denote by $v_{ij} = q^{\frac{i-j}{n}}$ for $i > j$ and $v_{ij} = q^{\frac{n+i-j}{n}}$ for $i < j$. Formulas (1.9.32)–(1.9.33) implies the following assertions. For $i > j$

$$\frac{\left(q^{1+\frac{i-j}{n}};q,q\right)_\infty}{\left(q^{l_i-l_j+1+\frac{i-j}{n}};q,q\right)_\infty} \times \frac{\left(q^{1+\frac{j-i}{n}};q,q\right)_\infty}{\left(q^{l_j-l_i+1+\frac{j-i}{n}};q,q\right)_\infty} =$$

$$(v_{ij};q)_\infty^{l_i-l_j}\,(v_{ji};q)_\infty^{l_j-l_i} \prod_{k=0}^{l_i-l_j-1} \frac{(-1)^{l_i-l_j-k}}{\left(q^{\frac{k}{2}+\frac{i-j}{2n}} - q^{-\frac{k}{2}-\frac{i-j}{2n}}\right)^{2(l_i-l_j-k)}} \tag{1.9.34}$$

For $j > i$

$$\frac{\left(q^{1+\frac{i-j}{n}};q,q\right)_\infty}{\left(q^{l_i-l_j+1+\frac{i-j}{n}};q,q\right)_\infty} \times \frac{\left(q^{1+\frac{j-i}{n}};q,q\right)_\infty}{\left(q^{l_j-l_i+1+\frac{j-i}{n}};q,q\right)_\infty} =$$

$$(v_{ij};q)_\infty^{l_i-l_j}\,(v_{ji};q)_\infty^{l_j-l_i} \prod_{k=1}^{l_i-l_j-1} \frac{(-1)^{l_i-l_j-k}}{\left(q^{\frac{k}{2}+\frac{i-j}{2n}} - q^{-\frac{k}{2}-\frac{i-j}{2n}}\right)^{2(l_i-l_j-k)}} \tag{1.9.35}$$

Using identities (1.9.34)–(1.9.35), we obtain

$$\prod_{i\neq j} \frac{\left(q^{1+\frac{i-j}{n}};q,q\right)_\infty}{\left(q^{l_i-l_j+1+\frac{i-j}{n}};q,q\right)_\infty} = \prod_{(i,j,k)} \frac{(-1)^{l_i-l_j-k}}{\left(q^{\frac{k}{2}+\frac{i-j}{2n}} - q^{-\frac{k}{2}-\frac{i-j}{2n}}\right)^{2(l_i-l_j-k)}}. \tag{1.9.36}$$

To finish the proof, it remains to clarify the sign. This product already appeared as the product over all hooks. For diagram λ the number of hooks is $|\lambda|$. □

Proof of Theorem 1.9.3. We will provide two different expressions for $\langle W(1|q^{1/n}), W(z^{1/n}|1)\rangle$ to prove the theorem. On one hand (by Proposition 1.9.10)

$$\left\langle W(1|q^{1/n}), W(z^{1/n}|1)\right\rangle = \left(q^{\frac{1}{n}}z^{\frac{1}{n}};q^{\frac{1}{n}},q^{\frac{1}{n}}\right)_\infty \tag{1.9.37}$$

On the other hand (by Theorem 1.9.2 and Proposition 1.9.13)

$$\left\langle W(1|q^{1/n}), W(z^{1/n}|1)\right\rangle = \sum_{(l_0,\dots,l_{n-1})\in\mathbf{Q}_{(n)}} \frac{z^{\frac{1}{2}\sum\left((l_i+\frac{i}{n})^2-(\frac{i}{n})^2\right)}}{\prod_{(i,j,k)}\left(q^{\frac{1}{2}(k+\frac{i-j}{n})}-q^{-\frac{1}{2}(k+\frac{i-j}{n})}\right)^{2(l_i-l_j-k)}}\times$$
$$(-1)^{|\lambda|}\left\langle W(1|q^{1-\frac{n-1}{n}-l_{n-1}},\dots,q^{1-l_0}), W(z|q^{l_0},\dots,q^{\frac{n-1}{n}+l_{n-1}})\right\rangle_{ss} \quad (1.9.38)$$

Note that to prove (1.9.38) we also used following observations: lengths of hooks in λ and λ' coincides, and $c(\lambda)+c(\lambda')=0$.

Multiplying RHS of (1.9.37) and (1.9.38) by $z^{\frac{1}{2}\sum\frac{i^2}{n^2}}\prod_{i\neq j}\frac{1}{\left(q^{1+\frac{i-j}{n}};q,q\right)_\infty}$ we obtain

$$z^{\frac{1}{2}\sum\frac{i^2}{n^2}}\prod_{i\neq j}\frac{1}{(q^{1+\frac{i-j}{n}};q,q)_\infty}\left(q^{\frac{1}{n}}z^{\frac{1}{n}};q^{\frac{1}{n}},q^{\frac{1}{n}}\right)_\infty = \sum_{(l_0,\dots,l_{n-1})\in\mathbf{Q}_{(n)}}\mathcal{Z}\left(q^{l_0},q^{\frac{1}{n}+l_1},\dots,q^{\frac{n-1}{n}+l_{n-1}};z\right) \quad (1.9.39)$$

Note that here we applied Proposition 1.9.14. □

1.10 Regular product

In this section, we develop general theory of regular product. Term 'regular product' should be considered as an opposite to regularized (i.e. normally ordered) product.

Let $A(z)=\sum_{k\in\mathbb{Z}}A_kz^{-k}$ be a formal power series with coefficients in $\mathrm{End}(V)$ for a vector space V.

Definition 1.10.1. *The series $A(z)$ is called smooth if for any vector $v\in V$ there exists N such that $A_kv=0$ for $k\geqslant N$.*

Let $G(z,w)=\sum_{k,l\in\mathbb{Z}}G_{k,l}z^{-k}w^{-l}$ be a formal power series in two variables with operator coefficients. The operators $G_{k,l}$ acts on a vector space V.

Definition 1.10.2. *We will call $G(z,w)$ regular if for any N and for any $v\in V$ there are only finitely many $G_{k,l}$ such that $k+l=N$ and $G_{k,l}v\neq 0$.*

If a current $G(z,w)$ in two variables is regular one can substitute $w=az$ and obtain well-defined power series $G(z,az)$ for any $a\in\mathbb{C}$.

Let $A(z)$ and $B(w)$ be two smooth formal power series with operator coefficients. Recall definition of normal ordering. Denote $A_+(z)=\sum_{k\geqslant 0}A_{-k}z^k$ and $A_-(z)=\sum_{k<0}A_{-k}z^k$.

Definition 1.10.3. *Normal ordered product is defined as* $:A(z)B(w):=A_+(z)B(w)+(-1)^\epsilon B(w)A_-(z)$

The sign $(-1)^\epsilon$ depends on *parity* of $A(z)$ and $B(z)$ in the standard way. Note that smooth formal power series in two variables $:A(z)B(w):$ is regular. Formal power series $A(z)$ and $B(z)$ are called local (in weaker sence) if

$$A(z)B(w)-(-1)^\epsilon B(w)A(z)=\sum_{i=1}^{N}\sum_{j=0}^{s_i}C_j^{(i)}(w)\ \partial_w^j\delta(a_iz,w) \quad (1.10.1)$$

where $s_1,\dots,s_N\in\mathbb{Z}_{\geq 0}$, $a_1,\dots a_N\in\mathbb{C}$ and $C_j^{(i)}(w)$ are operator valued power series.

Then one has the following OPE

$$A(z)B(w)=\sum\frac{C_j^{(i)}(w)}{(a_iz)^j\left(1-\frac{w}{a_iz}\right)^j}+:A(z)B(w): \quad (1.10.2)$$

Proposition 1.10.1. *If currents $A(z)$ and $B(w)$ are smooth and satisfy* (1.10.1)*, then the following product $(a_1z-w)^{s_1}\cdots(a_Nz-w)^{s_N}A(z)B(w)$ is regular.*

Definition 1.10.4. *For $a \neq a_i$ define regular product*

$$A(z)B(az) := \frac{\Big((a_1z-w)^{s_1}\cdots(a_Nz-w)^{s_N}A(z)B(w)\Big)\Big|_{w=az}}{(a_1z-az)^{s_1}\cdots(a_Nz-az)^{s_N}} \tag{1.10.3}$$

From (1.10.2) one obtains that normally ordered product and regular product are connected by the following relation

$$A(z)B(az) = \sum \frac{C_j^{(i)}(az)}{(a_iz)^j(1-a/a_i)^j} + :A(z)B(az): \tag{1.10.4}$$

Example 1.10.1. Let us consider case of fermions $A(z) = \psi(z)$, $B(z) = \psi^*(z)$, introduced in Section 1.3.2. Beware, that we use notation $A(z) = A_k z^{-k}$, but $\psi(z) = \psi_i z^{-i-1}$ (hence $A_k = \psi_{k-1}$). Comparing formulas (1.3.10)–(1.3.11) with Definition 1.10.3, we conclude

$$:\psi(z)\psi^*(w): =: \psi(z)\psi^*(w) :_{(0)} \tag{1.10.5}$$

Using l-depended normal ordering, we obtain

$$\psi(z)\psi^*(w) = \frac{w^l z^{-l-1}}{1-w/z} + :\psi(z)\psi^*(w):_{(l)} \tag{1.10.6}$$

Hence

$$\psi(z)\psi^*(qz) = \frac{q^l}{1-q}z^{-1} + :\psi(z)\psi(qz):_{(l)} \tag{1.10.7}$$

This relation was used in formula (1.3.13).

Example 1.10.2. Let $A(z) = B(z) = E(z)$. Then

$$E(z)E(w) = \frac{q^{-1}w}{z-q^{-1}w}E_2(q^{-1}w) - \frac{qw}{z-qw}E_2(w) + :E(z)E(w): \tag{1.10.8}$$

Therefore,

$$E^2(w) = \frac{1}{q-1}E_2(q^{-1}w) + \frac{q}{q-1}E_2(w) + :E(w)E(w): \tag{1.10.9}$$

Let us comment on deep meaning of formula (1.10.9). One can present algebra $\mathfrak{Diff}_q$ using currents $E_k(z)$ (currents of Lie algebra type) or $E^k(z)$ (currents of q-W algebra type). This two series of currents are connected in non-trivial way starting from $k=2$. For $k=2$ they are related by (1.10.9). For general k see formula (7.17) in [Neg18].

Proposition 1.10.2. *Regular product is (super) commutative and associative*

$$A(z)B(az) = (-1)^{\epsilon}B(az)A(z) \tag{1.10.10}$$
$$A(a_1z)\left(B(a_2z)C(a_3z)\right) = \left(A(a_1z)B(a_2z)\right)C(a_3z) \tag{1.10.11}$$

(of course we assume that these regular products are well defined).

Proposition 1.10.3. *Let $A(z)$, $B(z)$, and $a \in \mathbb{C}$ be as in Definition 1.10.4.Then*

$$\partial_z \left(A(z)B(az)\right) = A'(z)B(az) + aA(z)B'(az) \tag{1.10.12}$$

Proof. Let $f(z,w)$ be a polynomial such that following power series in two variables are regular

$$\left(\partial_z f(z,w)\right) A(z)B(w), \quad \left(\partial_w f(z,w)\right) A(z)B(w), \quad f(z,w)\left(\partial_z A(z)\right) B(w), \quad f(z,w)A(z)\left(\partial_w B(w)\right). \tag{1.10.13}$$

Moreover, assume that $f(z,az) \neq 0$. It is easy to see that such $f(z,w)$ exists.

$$\partial_z \left(A(z)B(az)\right) = \left(\partial_z + a\partial_w\right) \frac{f(z,w)A(z)B(w)}{f(z,az)}\Bigg|_{w=az}$$

One should differentiate this expression by application of Leibniz rule (and obtain six summands). Due to our assumptions, each of these summands is regular in the sense of Definition 1.10.2. Hence, one can substitute $w = az$ to each summand separately. The proof is finished by straightforward computation. □

As a corollary we prove formula (1.7.28).

Proof of (1.7.28).

$$\partial_z \mathbb{E}^{(k+1)}(z,w)\Big|_{z=w} = (z-qw)(z-q^{-1}w)E'(z)E^k(w) + (2z - qw - q^{-1}w)E(z)E^k(w)\Big|_{z=w}$$

$$\frac{(z-qw)^2(z-q^{-1}w)^2}{(1-q)(1-q^{-1})z^2}E'(z)E^k(w) + \frac{(z-qw)(z-q^{-1}w)}{(1-q)(1-q^{-1})z^2}(2z-qw-q^{-1}w)E(z)E^k(w)\Bigg|_{z=w}$$

Note that each summand is regular. Hence, we are allowed substitute $z = w$ to each of them separately

$$\partial_z \mathbb{E}^{(k+1)}(z,w)\Big|_{z=w} = (1-q)(1-q^{-1})w^2E'(w)E^k(w) + (2-q-q^{-1})wE^{k+1}(w) \tag{1.10.14}$$

Using Propositions 1.10.2 and 1.10.3, we can prove inductively that $\partial_w E^{k+1}(w) = (k+1)E'(w)E^k(w)$. To finish the proof, we substitute last formula into (1.10.14). □

1.11 Serre relation

This Section is devoted to detailed study of Serre relation.

$$z_2 z_3^{-1}[E(z_1),[E(z_2),E(z_3)]] + \text{cyclic} = 0. \tag{1.11.1}$$

Let $\bar{E}(z)$ be a current satisfying

$$(z-qw)(z-q^{-1}w)[\bar{E}(z),\bar{E}(w)] = 0. \tag{1.11.2}$$

Remark 1.11.1. Let us emphasize the difference between $E(z)$ and $\bar{E}(z)$. Current $E(z)$ is a current from $\mathfrak{Diff}_q$, but current $\bar{E}(z)$ is just a current satisfying (1.11.2). We need $\bar{E}(z)$ to formulate equivalent conditions to Serre relations.

Define $\delta(z_1/z_2/z_3) = \sum_{a+b+c=0} z_1^a z_2^b z_3^c$ for $a,b,c \in \mathbb{Z}$.

Proposition 1.11.1. *There exist three currents $R_1(z)$, $R_2(z)$ and $\bar{E}_3(z)$ such that triple commutator $[\bar{E}(z_1),[\bar{E}(z_2),\bar{E}(z_3)]]$ equals to*

$$\bar{E}_3(z_1)\delta(q^2z_1/qz_2/z_3) - \bar{E}_3(z_2)\delta(q^2z_2/qz_3/z_1) - \bar{E}_3(z_1)\delta(q^2z_1/qz_3/z_2) + \bar{E}_3(z_3)\delta(q^2z_3/qz_2/z_1) +$$
$$R_1(z_1)\delta(z_1/qz_2/z_3) - R_1(z_1)\delta(z_1/z_2/qz_3) + R_2(z_1)\delta(z_1/z_2/q^{-1}z_3) - R_2(z_1)(z_1/q^{-1}z_2/z_3). \tag{1.11.3}$$

Proof. First of all, note that condition (1.11.2) is equivalent to existence of currents $\bar{E}_2(z)$ and $\bar{E}_2^\circ(w)$ such that

$$[\bar{E}(z), \bar{E}(w)] = \bar{E}_2(z)\delta(w/qz) - \bar{E}_2^\circ(w)\delta(z/qw). \tag{1.11.4}$$

Commutator $[\bar{E}(z), \bar{E}(w)]$ is skew symmetric on z and w, hence $\bar{E}_2(z) = \bar{E}_2^\circ(z)$ Now consider triple commutator $[\bar{E}(z_1), [\bar{E}(z_2), \bar{E}(z_3)]$. Jacobi identity and relation (1.11.2) imply

$$(z_1 - qz_2)(z_1 - q^{-1}z_2)(z_1 - qz_3)(z_1 - q^{-1}z_3)[\bar{E}(z_1), [\bar{E}(z_2), \bar{E}(z_3)] = 0 \tag{1.11.5}$$

Substituting (1.11.4) to (1.11.5), we obtain

$$\begin{aligned}(z_1 - qz_2)(z_1 - q^{-1}z_2)(z_1 - q^2z_2)(z_1 - z_2)[\bar{E}(z_1), \bar{E}_2(z_2)]\delta(z_3/qz_2)+ \\ (z_1 - q^2z_3)(z_1 - z_3)(z_1 - qz_3)(z_1 - q^{-1}z_3)[\bar{E}(z_1), \bar{E}_2(z_3)]\delta(z_2/qz_3) = 0\end{aligned} \tag{1.11.6}$$

This implies that $\left[\bar{E}(z_1)[\bar{E}(z_2), \bar{E}(z_3)]\right]$ is indeed a sum of triple delta functions with some operator coefficients as in (1.11.3); it remains to prove proposed relations on the coefficients.

Note that triple commutator $\left[\bar{E}(z_1), [\bar{E}(z_2), \bar{E}(z_3)]\right]$ is skew symmetric on z_2, z_3. Also note that the sum over cyclic permutations is zero. This implies relation (1.11.3). □

Proposition 1.11.2. *Serre relation for $\bar{E}(z)$ is equivalent to $R_1(z) = R_2(z) = 0$.*

Proof. Straightforward computation. □

1.11.1 Operator product expansion for $E(w_1)\cdots E(w_k)$

One can find reformulation of Serre relation in terms of OPE in [FJMM16, Section 3.3]

Proposition 1.11.3. *Formal power series in three variables $\bar{E}(z_1)\bar{E}(z_2)\bar{E}(z_3)$ can be presented as sum some of regular part and singular part. Regular part is some regular power series in three variables, singular part has a form*

$$\begin{aligned}\sum_{\epsilon_1,\epsilon_2=\pm1} \Big((1 - q^{\epsilon_1}\frac{z_2}{z_1})^{-1}(1 - q^{\epsilon_2}\frac{z_3}{z_1})^{-1}R^{(1)}_{\epsilon_1,\epsilon_2}(z_3) + (1 - q^{\epsilon_1}\frac{z_2}{z_1})^{-1}(1 - q^{\epsilon_2}\frac{z_3}{z_2})^{-1}R^{(2)}_{\epsilon_1,\epsilon_2}(z_3) + \\ (1 - q^{\epsilon_1}\frac{z_3}{z_1})^{-1}(1 - q^{\epsilon_2}\frac{z_3}{z_2})^{-1}R^{(3)}_{\epsilon_1,\epsilon_2}(z_3)\Big) + \sum_{\epsilon=\pm1}\Big((1 - q^{\epsilon}\frac{z_3}{z_2})^{-1}R^{(1)}_{\epsilon}(z_1, z_3) + \\ (1 - q^{\epsilon}\frac{z_3}{z_1})^{-1}R^{(2)}_{\epsilon}(z_2, z_3) + (1 - q^{\epsilon}\frac{z_2}{z_1})^{-1}R^{(3)}_{\epsilon}(z_1, z_3)\Big) + \mathrm{reg}\end{aligned} \tag{1.11.7}$$

Proof. Denote $G(z_1, z_2, z_3) := \prod_{i<j}(z_i - qz_j)(z_i - q^{-1}z_j)\ E(z_1)E(z_2)E(z_3)$. Relation (1.11.2) yields $G(z_1, z_2, z_3)$ to be regular. □

Proposition 1.11.4. *Serre relation for $\bar{E}(z)$ is equivalent to condition that singular part of $E(z_1)E(z_2)E(z_3)$ restricted to $z_1 = z_3$ has no poles of order greater than 1.*

Proof. Note that second order pole can appear only from terms of form

$$\left(1 - q^{\epsilon}\frac{z_2}{z_1}\right)^{-1}\left(1 - q^{-\epsilon}\frac{z_3}{z_2}\right)^{-1} R^{(2)}_{\epsilon,-\epsilon}(z_3)$$

for $\epsilon = \pm1$. Note that these two poles can not cancel because they are at the different points $z_1 = q^{\epsilon}z_2$. On the other hand, $R_1(z) = R^{(2)}_{1,-1}(z)$ and $R_2(z) = R^{(2)}_{-1,1}(z)$. Application of Proposition 1.11.2 completes the proof. □

Corollary 1.11.5. *$E(z)E^k(w)$ has no poles of order greater than one. Poles may appear only at points $z = q^{\pm1}w$.*

Proof. To study $E(z)E(w)^k$ we will consider OPE $E(z)E(w_1)\cdots E(w_k)$ and substitute $w_i = w$. Only term $(z-q^\epsilon w_i)^{-1}(z-q^\epsilon w_j)^{-1}\dots$ can give poles of order higher than 1 after substitution. OPE is symmetric on $z, w_1, \dots, w_k$ as a rational function. We will consider order $\big(E(w_i)E(z)E(w_j)\big)\cdots$. According to Proposition 1.11.4, the term $(z-q^\epsilon w_i)^{-1}(z-q^\epsilon w_j)^{-1}\dots$ does not appear. □

1.12 Homomorphism from $\mathfrak{Diff}_q$ to W-algebra

This Section is devoted to proof of Propositions 1.7.8. The proof is a straightforward check of relation from Proposition 1.2.2. The relations will be checked for operators

$$\tilde{H}(z) = \sum_{j\neq 0} \tilde{H}_j z^{-j}, \quad c = n, \quad c' = n_{tw}, \tag{1.12.1}$$

$$\tilde{E}(z) = \mu \exp\left(\frac{1}{n}\tilde{\varphi}_-(z)\right) T_1(z) \exp\left(\frac{1}{n}\tilde{\varphi}_+(z)\right), \tag{1.12.2}$$

$$\tilde{F}(z) = -\frac{\mu^{-1}z^{-n_{tw}}}{(q^{\frac12}-q^{-\frac12})^2} \exp\left(-\frac{1}{n}\tilde{\varphi}_-(z)\right) T_{n-1}(z) \exp\left(-\frac{1}{n}\tilde{\varphi}_+(z)\right). \tag{1.12.3}$$

Proposition 1.12.1. *Relations* (1.2.8), (1.2.9) *are satisfied.*

Proof. Straightforward. □

Proposition 1.12.2. *Currents* $\tilde{E}(z)$ *and* $\tilde{F}(z)$ *satisfy relation* (1.2.10).

Proof. It is easy to see that

$$\tilde{E}(z)\tilde{E}(w) = \frac{(1-\frac{w}{z})^2}{(1-q\frac{w}{z})(1-q^{-1}\frac{w}{z})} f_{1,n}(w/z)\ \mu^2$$
$$\exp\left(\frac{1}{n}(\tilde{\varphi}_-(z)+\tilde{\varphi}_-(w))\right) T_1(z)T_1(w) \exp\left(\frac{1}{n}(\tilde{\varphi}_+(z)+\tilde{\varphi}_+(w))\right) \tag{1.12.4}$$

Thus,

$$(z-qw)(z-q^{-1}w)\left(\tilde{E}(z)\tilde{E}(w)-\tilde{E}(w)\tilde{E}(z)\right) = (z-w)^2 \exp\left(\frac{1}{n}(\tilde{\varphi}_-(z)+\tilde{\varphi}_-(w))\right)$$
$$(f_{1,n}(w/z)T_1(z)T_1(w) - f_{1,n}(z/w)T_1(w)T_1(z)) \exp\left(\frac{1}{n}(\tilde{\varphi}_+(z)+\tilde{\varphi}_+(w))\right) = 0$$

Proof for $\tilde{F}(z)$ is analogous. □

Proposition 1.12.3. *Currents* $\tilde{E}(z)$ *and* $\tilde{F}(z)$ *satisfy relation* (1.2.11) *for* $c = n$ *and* $c' = n_{tw}$.

Proof. It is easy to see that

$$\tilde{E}(z)\tilde{F}(w) = -\frac{1}{(q^{\frac12}-q^{-\frac12})^2} f_{n-1,n}(w/z)$$
$$\exp\left(\frac{1}{n}(\tilde{\varphi}_-(z)-\tilde{\varphi}_-(w))\right) T_1(z)T_{n-1}(w) \exp\left(\frac{1}{n}(\tilde{\varphi}_+(z)-\tilde{\varphi}_+(w))\right) w^{-n_{tw}}$$

Thus,

$$\left[\tilde{E}(z), \tilde{F}(w)\right] =$$
$$\exp\left(\frac{1}{n}\left(\tilde{\varphi}_-(z)-\tilde{\varphi}_-(w)\right)\right)\left(n\frac{w}{z}\delta'\left(\frac{w}{z}\right) w^{n_{tw}} + w\delta\left(\frac{w}{z}\right)\partial_w w^{n_{tw}}\right) \exp\left(\frac{1}{n}\left(\tilde{\varphi}_+(z)-\tilde{\varphi}_+(w)\right)\right) w^{-n_{tw}}$$

Consequently,

$$[\tilde{E}(z), \tilde{F}(w)] = n\frac{w}{z}\delta'\left(\frac{w}{z}\right) + \left(\tilde{H}(q^{-\frac{1}{2}}z) - \tilde{H}(q^{\frac{1}{2}}z) + n_{tw}\right)\delta\left(\frac{w}{z}\right)$$

□

Denote by $\tilde{\mathbb{E}}^{(2)}(z,w) = (z-qw)(z-q^{-1}w)\tilde{E}(z)\tilde{E}(w)$.

Lemma 1.12.4. $\tilde{E}^2(z) = 2\mu^2 \exp\left(\frac{2}{n}\tilde{\varphi}_-(z)\right) T_2(z) \exp\left(\frac{2}{n}\tilde{\varphi}_+(z)\right)$

Proof.

$$f_{1,n}(w/z)T_1(z)T_1(w) - f_{1,n}(z/w)T_1(w)T_1(z) =$$
$$\mu^{-2}\exp\left(-\frac{1}{n}\left(\tilde{\varphi}_-(z)+\tilde{\varphi}_-(w)\right)\right)\ \tilde{\mathbb{E}}^{(2)}(z,w)\ \exp\left(-\frac{1}{n}\left(\tilde{\varphi}_+(z)+\tilde{\varphi}_+(w)\right)\right)\ \partial_w\left(w^{-1}\delta(w/z)\right) =$$
$$-\mu^{-2}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2\exp\left(-\frac{2}{n}\tilde{\varphi}_-(w)\right)\tilde{E}^2(w)\exp\left(-\frac{2}{n}\tilde{\varphi}_+(w)\right)\frac{w}{z}\delta'(w/z) + \cdots\delta(w/z)$$

On the other hand, LHS can be found from relation (1.7.13). Comparing coefficients of $\delta'(w/z)$ completes the proof. □

Proposition 1.12.5. *$\tilde{E}(z)$ and $\tilde{F}(z)$ satisfy Serre relation* (1.2.12).

Proof. Using Lemma 1.12.4, we see that

$$\tilde{E}(z)\tilde{E}^2(w) = 2\mu^3\frac{(1-\frac{w}{z})^2}{(1-q\frac{w}{z})(1-q^{-1}\frac{w}{z})}$$
$$\exp\left(\frac{1}{n}\left(\tilde{\varphi}_-(z)+2\tilde{\varphi}_-(w)\right)\right) f_{2,n}(w/z)T_1(z)T_2(w)\ \exp\left(\frac{1}{n}\left(\tilde{\varphi}_+(z)+2\tilde{\varphi}_+(w)\right)\right) \quad (1.12.5)$$

Note that $(z-w)^2 f_{2,n}(w/z)T_1(z)T_2(w)$ is regular. Proposition 1.11.4 completes the proof of Serre relation for $\tilde{E}(z)$. Proof for $\tilde{F}(z)$ is analogous. □

1.13 Whittaker vector

1.13.1 Uniqueness of Whittaker vector

Recall that operators $E^k[d]$ are defined by $E^k(z) = \sum_d E^k[d]z^{-d}$.

Proposition 1.13.1. *Whittaker vector $W(z|u_1,\dots,u_n)$ is annihilated by $E^k[d]$ for $d>0$ and $k = 1,\dots,n-1$.*

Proof. Actually we will prove that Whittaker vector is annihilated by $E^k[d]$ for $nd > k$. To do this we will need [Neg18, eq. (7.17)]. Let us rewrite this with respect to our notation

$$E^k[d] = \sum_{t\geqslant 1}(-1)^{k-t}\sum_{v=\{\frac{d_1}{k_1}\leqslant\frac{d_2}{k_2}\leqslant\cdots\leqslant\frac{d_t}{k_t}\}}^{\substack{k_i\in\mathbb{N},\ d_i\in\mathbb{Z};\\ k_1+\cdots+k_t=k\\ d_1+\cdots+d_t=d}} c_v E_{k_1,d_1}\cdots E_{k_t,d_t}. \quad (1.13.1)$$

Here c_v denotes some combinatorially defined coefficient, which is not quite important for us. Inequality $nd > k$ implies $nd_t > k_t$; therefore $E_{k_t,d_t}W(z|u_1,\dots,u_n) = 0$. So, any summand of RHS annihilates Whittaker vector. □

Let us denote $\mathcal{W}_q(\mathfrak{gl}_n) := \mathcal{W}_q(\mathfrak{gl}_n, 0) = U(\mathfrak{Diff}_q)/J_{n,0}$. Denote Verma module for $\mathcal{W}_q(\mathfrak{gl}_n)$ by $\mathcal{V}^{\mathcal{W}_q(\mathfrak{gl}_n)}_{\lambda_1,\dots,\lambda_d}$ (cf. Definition 1.7.4).

Definition 1.13.1. *For each graded $\mathfrak{Diff}_q$ module M let us define Shapovalov dual module $M^\vee$. As a vector space $M^\vee$ is graded dual to M. Action is defined by requirement that canonical pairing $M^\vee \otimes M \to \mathbb{C}$ is Shapovalov.*

Finally note that involution $E_{a,b} \mapsto E_{-a,-b}$ maps ideal $J_{n,0}$ to $J_{n,0}$ (maybe with different μ). Hence if M is a $\mathcal{W}_q(\mathfrak{gl}_n)$-module then so is $M^\vee$.

Proposition 1.13.2. *Let $u_i/u_j \neq q^k$ for any $k \in \mathbb{Z}$ (cf. Lemma 1.9.1). There is no more than one Whittaker vector $W(z|u_1,\dots,u_N) \in \mathcal{F}_{u_1} \otimes \dots \otimes \mathcal{F}_{u_N}$.*

Proof. Denote by $\mathfrak{n}$ a subalgebra of $\mathcal{W}_q(\mathfrak{gl}_n)$ generated by $E^k[d]$ and H_j for $k = 1,\dots,n-1$, $d > 0$ and $j > 0$. Analogously, let $\mathfrak{n}^\vee$ be a subalgebra of $\mathcal{W}_q(\mathfrak{gl}_n)$ generated by $F^k[-d]$ and H_{-j} for $k = 1,\dots,n-1$, $d > 0$ and $j > 0$. Note that involution $E_{a,b} \mapsto E_{-a,-b}$ induces an involoution on $\mathcal{W}_q(\mathfrak{gl}_n)$ which swaps $\mathfrak{n}$ and $\mathfrak{n}^\vee$.

Consider $\mathcal{F}_{u_1} \otimes \dots \otimes \mathcal{F}_{u_n}$ as a $\mathcal{W}_q(\mathfrak{gl}_n)$-module. Whittaker vector is an eigenvector for $\mathfrak{n}$. Hence, it is enough to show that $\mathcal{F}_{u_1} \otimes \dots \otimes \mathcal{F}_{u_n}$ is cocyclic for $\mathfrak{n}$. Equivalently, we need to prove that Shapovalov dual module $(\mathcal{F}_{u_1} \otimes \dots \otimes \mathcal{F}_{u_n})^\vee \cong \mathcal{F}_{q/u_n} \otimes \dots \otimes \mathcal{F}_{q/u_1}$ is cyclic for $\mathfrak{n}^\vee$.

Fock module $\mathcal{F}_{q/u_n} \otimes \dots \otimes \mathcal{F}_{q/u_1}$ is isomorphic to Verma module $\mathcal{V}^{\mathcal{W}_q(\mathfrak{gl}_n)}_{\lambda_1,\dots,\lambda_d}$ for corresponding λ_j by Theorem 1.7.3. Verma module $\mathcal{V}^{\mathcal{W}_q(\mathfrak{gl}_n)}_{\lambda_1,\dots,\lambda_d}$ is cyclic for $\mathfrak{n}^\vee$ by an analogue of Proposition 1.7.15 for $F^k[d]$. □

1.13.2 Construction of Whittaker vector

Let (n_1', n_1) and (n_2', n_2) be a basis of $\mathbb{Z}^2$.

Theorem 1.13.1 ([AFS12])**.** *There exist homomorphisms*

$$\Phi\colon \mathcal{F}_u^{(n_1',n_1)} \otimes \mathcal{F}_v^{(n_2',n_2)} \to \mathcal{F}_{-q^{-1/2}uv}^{(n_1'+n_2',n_1+n_2)}, \tag{1.13.2}$$

$$\Phi^*\colon \mathcal{F}_{-q^{-1/2}uv}^{(n_1'+n_2',n_1+n_2)} \to \mathcal{F}_u^{(n_1',n_1)} \otimes \mathcal{F}_v^{(n_2',n_2)}. \tag{1.13.3}$$

These homomorphisms are defined uniquely up to normalization.

Remark 1.13.1. Actually operators Φ and Φ^* maps to graded completion of $\mathcal{F}_{-q^{-1/2}uv}^{(n_1'+n_2',n_1+n_2)}$ and $\mathcal{F}_u^{(n_1',n_1)} \otimes \mathcal{F}_v^{(n_2',n_2)}$ correspondingly. Abusing notation, we will use the same symbol for a module and its completion. Moreover, we are going to consider a composition of such Φ^*; there appear an infinite sum as a result of such composition (a priori this sum does not make sense). We will use a calculus approach to infinite sums; below we will provide a sufficient condition for convergence of the series.

Denote by $\Phi^*_\mu(u)$ component of Φ^* corresponding to $|\mu\rangle \in \mathcal{F}_u^{(n_2',n_2)}$. More precisely, for any x in $\mathcal{F}_{-q^{-1/2}uv}^{(n_1'+n_2',n_1+n_2)}$.

$$\Phi^* \cdot x = \sum_\mu |\mu\rangle \otimes \left(\Phi^*_\mu(u) \cdot x\right) \tag{1.13.4}$$

To simplify our notation we will consider particular case

$$\Phi^*\colon \mathcal{F}_{-q^{-1/2}uv}^{(1,-k)} \to \mathcal{F}_u^{(0,1)} \otimes \mathcal{F}_v^{(1,-k-1)} \tag{1.13.5}$$

Note that both $\mathcal{F}_{-q^{-1/2}uv}^{(1,-k)}$ and $\mathcal{F}_v^{(1,-k-1)}$ are Fock modules for Heisenberg algebra generated by $a_k = E_{k,0}$.

Proposition 1.13.3 ([AFS12]). *Operator $\Phi^*_\mu(u)$ is defined by following explicit formulas*

$$\Phi^*_\varnothing(u) = :\exp\left(\sum_{k\neq 0}\frac{u^{-k}}{k(1-q^{-k})}a_k\right):, \tag{1.13.6}$$

$$\Phi^*_\mu(u) \sim :\Phi^*_\varnothing(u)\prod_{s\in\lambda}F(q^{c(s)-\frac{1}{2}}u):. \tag{1.13.7}$$

Here sign $\sim$ means up to multiplication by a number. Recall

$$F(z) \sim :\exp\left(\sum_k \frac{q^{k/2}-q^{-k/2}}{k}a_k z^{-k}\right):. \tag{1.13.8}$$

Corollary 1.13.4. *Heisenberg normal ordering is given by*

$$\Phi^*_\lambda(u)\Phi^*_\mu(v) = \frac{f_{\lambda,\mu}(v/u)}{(qv/u;q,q)_\infty} :\Phi^*_\lambda(u)\Phi^*_\mu(v): \tag{1.13.9}$$

for some rational function $f_{\lambda,\mu}(v/u) = \frac{\prod_i(1-q^{k_i}v/u)}{\prod_j(1-q^{l_j}v/u)}$; here l_j and k_i are integer numbers.

Consider a homomorphism

$$\tilde{\Phi}_0\colon \mathcal{F}^{(1,0)}_{q^{1/2}z^{-1}} \otimes \mathcal{F}^{(-1,n)}_{q^{1/2}z(-q^{-1/2})^n u_1\cdots u_n} \to \left(\mathcal{F}^{(0,1)}_{u_1}\otimes\dots\otimes\mathcal{F}^{(0,1)}_{u_n}\right)\otimes\mathcal{F}^{(1,-n)}_{q^{1/2}(-q^{1/2})^n(zu_1\dots u_n)^{-1}}\otimes\mathcal{F}^{(-1,n)}_{q^{1/2}z(-q^{-1/2})^n u_1\cdots u_n}. \tag{1.13.10}$$

obtained as composition of

$$\mathrm{id}^{\otimes k}\otimes\Phi^*\otimes\mathrm{id}\colon \left(\mathcal{F}^{(0,1)}_{u_1}\otimes\dots\otimes\mathcal{F}^{(0,1)}_{u_k}\right)\otimes\mathcal{F}^{(1,-k)}_{q^{1/2}(-q^{1/2})^k(zu_1\dots u_k)^{-1}}\otimes\mathcal{F}^{(-1,n)}_{q^{1/2}z(-q^{-1/2})^n u_1\cdots u_n} \to \left(\mathcal{F}^{(0,1)}_{u_1}\otimes\dots\otimes\mathcal{F}^{(0,1)}_{u_{k+1}}\right)\otimes\mathcal{F}^{(1,-k-1)}_{q^{1/2}(-q^{1/2})^{k+1}(zu_1\dots u_{k+1})^{-1}}\otimes\mathcal{F}^{(-1,n)}_{q^{1/2}z(-q^{-1/2})^n u_1\cdots u_n} \to \tag{1.13.11}$$

Lemma 1.13.5. *There exists a unique invariant pairing $\mathcal{F}_u\otimes\mathcal{F}_{qu^{-1}}\to\mathbb{C}$ such that $\langle 0|0\rangle = 1$.*

Proof. This is equivalent to Proposition 1.9.9. □

Composition of $\tilde{\Phi}_0$ and the pairing gives a homomorphism

$$\tilde{\Phi}_1\colon \mathcal{F}^{(1,0)}_{q^{\frac{1}{2}}z^{-1}}\otimes\mathcal{F}^{(-1,n)}_{q^{\frac{1}{2}}u_1\cdots u_n z(-q^{-1/2})^n} \to \mathcal{F}^{(0,1)}_{u_1}\otimes\dots\otimes\mathcal{F}^{(0,1)}_{u_n}. \tag{1.13.12}$$

Let us reformulate above inductive procedure via an explicit formula

$$\tilde{\Phi}_1\left(|\lambda_1\rangle\otimes|\lambda_2\rangle\right) = \sum\langle\lambda_2|\Phi^*_{\mu_n}(u_n)\dots\Phi^*_{\mu_1}(u_1)|\lambda_1\rangle\,|\mu_1\rangle\otimes\dots\otimes|\mu_n\rangle. \tag{1.13.13}$$

As we warned in Remark 1.13.1, operator $\tilde{\Phi}_1$ is not a priori well defined. However, the series (appearing from the composition) converges in a domain $|u_1|\ll|u_2|\ll\dots\ll|u_n|$. This assertion follows from a formula

$$\langle\lambda_2|\Phi^*_{\mu_n}(u_n)\dots\Phi^*_{\mu_1}(u_1)|\lambda_1\rangle = \prod_{i<j}\frac{f_{\mu_i,\mu_j}(u_i/u_j)}{(qu_i/u_j;q,q)_\infty}\langle\lambda_1|:\Phi^*_{\mu_n}(u_n)\dots\Phi^*_{\mu_1}(u_1):|\lambda_2\rangle \tag{1.13.14}$$

Moreover, one can consider analytic continuation of obtained function given by RHS of the formula. Corollary 1.13.4 implies that we can extend the domain to $u_i/u_j\neq q^k$ for any $k\in\mathbb{Z}$. Evidently, analytic continuation also enjoys intertwiner property. Hence we obtained following proposition

Proposition 1.13.6. *If $u_i/u_j \neq q^k$ for any $k \in \mathbb{Z}$, then there is an intertwiner*

$$\tilde{\Phi} : \mathcal{F}^{(1,0)}_{q^{1/2}z^{-1}} \otimes \mathcal{F}^{(-1,n)}_{q^{1/2}u_1\cdots u_n z(-q^{-1/2})^n} \to \mathcal{F}^{(0,1)}_{u_1} \otimes \cdots \otimes \mathcal{F}^{(0,1)}_{u_n} \tag{1.13.15}$$

Denote the highest vector of $\mathcal{F}^{(n_1,n_2)}_u$ by $|n_1, n_2\rangle$.

Theorem 1.13.2. *Whittaker vector $W(z|u_1,\dots,u_n) \in \mathcal{F}^{(0,1)}_{u_1} \otimes \cdots \otimes \mathcal{F}^{(0,1)}_{u_n}$ can be constructed via homomorphism $\tilde{\Phi}$ (as in (1.13.15)).*

$$W(z|u_1,\dots,u_n) := \prod_{i<j} (qu_i/u_j;q,q)_\infty \, \tilde{\Phi}\left(|1,0\rangle \otimes |-1,n\rangle\right) \tag{1.13.16}$$

Proof. Follows from (1.3.22). □

Remark 1.13.2. Recall that existence of Whittaker vector can be seen from geometric construction (see [Neg15a] and [Tsy17]).

Proof of Theorem 1.9.1. Existence and uniqueness follows from Theorem 1.13.2 and Proposition 1.13.2 correspondingly. □

1.13.3 Whittaker vector for $\mathcal{W}_q(\mathfrak{sl}_n)$ algebra

Let us define coefficient $\bar{c}_{1/m}$ by (cf. (1.13.1))

$$E^m[1] = \bar{c}_{1/m} E_{m,1} + \dots. \tag{1.13.17}$$

Also recall that for $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_n}$

$$\mu = \frac{1}{1-q}(u_1\cdots u_n)^{\frac{1}{n}}. \tag{1.13.18}$$

Definition 1.13.2. *For $m = 1,\dots n-1$ Whittaker vector $W^{\mathfrak{sl}_n}_m(z|u_1,\dots,u_n) \in \mathcal{F}^{\mathcal{W}_q(\mathfrak{sl}_n)}_{u_1,\dots,u_d}$ with respect to $\mathcal{W}_q(\mathfrak{sl}_n)$ is an eigenvector for $T_k[r]$ (for $k = 1,\dots,n-1$ and $r \geqslant 0$) with eigenvalues given by*

$$T_m[1]\; W^{\mathfrak{sl}_n}_m(z|u_1,\dots,u_n) = \frac{(-q^{\frac{1}{2}})^n u_1\cdots u_n z}{q^{-1/2}-q^{1/2}} \frac{\bar{c}_{1/m}\,\mu^{-m}}{m!} W^{\mathfrak{sl}_n}_m(z|u_1,\dots,u_n) \tag{1.13.19}$$

$$T_k[1]\; W^{\mathfrak{sl}_n}_m(z|u_1,\dots,u_n) = 0 \qquad \textit{for } k \neq m \tag{1.13.20}$$

$$T_k[r]\; W^{\mathfrak{sl}_n}_m(z|u_1,\dots,u_n) = 0 \qquad \textit{for } r \geqslant 2 \tag{1.13.21}$$

We require $W^{\mathfrak{sl}_n}_m(z|u_1,\dots,u_n) = |\bar{u}\rangle + \cdots$ to fix normalization (by dots we mean lower vectors).

One can find notion of Whittaker vector for $\mathcal{W}_q(\mathfrak{sl}_n)$ in the literature (see [Tak10]). In this section we will explain connection between notion of Whittaker vector $W^{\mathfrak{sl}_n}_m(z|u_1,\dots,u_n)$ and Whittaker vector $W(z|u_1,\dots,u_n)$ for $\mathfrak{Diff}_q$ (see Definition 1.9.1). Our plan to explain this connection is as follows. First we define Whittaker vector with respect to $\mathcal{W}_q(\mathfrak{gl}_n)$ (we denote it by $W^{\mathfrak{gl}_n}_m(z|u_1,\dots,u_n)$). Then we will see, that on the one hand, the vector $W^{\mathfrak{gl}_n}_m(z|u_1,\dots,u_n)$ is connected with $W^{\mathfrak{sl}_n}_m(z|u_1,\dots,u_n)$; on the other hand it is connected with $W(z|u_1,\dots,u_n)$.

Recall, that $\mathcal{W}_q(\mathfrak{gl}_n) = \mathfrak{Diff}_q/J_{n,0}$; ideal $J_{n,0}$ annihilates $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_n}$. Hence $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_n}$ is a representation of $\mathcal{W}_q(\mathfrak{gl}_n)$.

Definition 1.13.3. *For $m = 1, \dots n-1$ Whittaker vector $W_m^{\mathfrak{gl}_n}(z|u_1, \dots, u_n)$ is a vector belonging to $\mathcal{F}_{u_1} \otimes \dots \otimes \mathcal{F}_{u_n}$ and satisfying following conditions*

$$H_k W_m^{\mathfrak{gl}_n}(z|u_1, \dots, u_n) = 0 \qquad \text{for } k > 0 \tag{1.13.22}$$

$$E^m[1]\; W_m^{\mathfrak{gl}_n}(z|u_1, \dots, u_n) = \frac{(-q^{\frac{1}{2}})^n u_1 \cdots u_n z}{q^{-1/2} - q^{1/2}} \bar{c}_{1/m}\, W_m^{\mathfrak{gl}_n}(z|u_1, \dots, u_n) \tag{1.13.23}$$

$$E^k[1]\; W_m^{\mathfrak{gl}_n}(z|u_1, \dots, u_n) = 0 \qquad \text{for } k < m \tag{1.13.24}$$

$$E^k[r]\; W_m^{\mathfrak{gl}_n}(z|u_1, \dots, u_n) = 0 \qquad \text{for } r \geqslant 2 \text{ and } k \leqslant m \tag{1.13.25}$$

$$F^k[r]\; W_m^{\mathfrak{gl}_n}(z|u_1, \dots, u_n) = 0 \qquad \text{for } r \geqslant 1 \text{ and } k < n-m \tag{1.13.26}$$

We require $W_m^{\mathfrak{gl}_n}(z|u_1, \dots, u_n) = |\bar{u}\rangle + \cdots$ to fix normalization (by dots we mean lower vectors).

Recall that $\mathcal{F}_{u_1} \otimes \dots \otimes \mathcal{F}_{u_n} \cong \mathcal{F}_{u_1,\dots u_n}^{\mathcal{W}_q(\mathfrak{sl}_n)} \otimes F^H$ with respect algebra identification $\mathcal{W}_q(\mathfrak{gl}_n) \cong \mathcal{W}_q(\mathfrak{sl}_n) \otimes U(\mathfrak{Heis})$.

Lemma 1.13.7. *Vector satisfies properties of $W_m^{\mathfrak{gl}_n}(z|u_1, \dots, u_n)$ iff it is $W_m^{\mathfrak{sl}_n}(z|u_1, \dots, u_n) \otimes |\varnothing\rangle_H$*

Proof. Note that

$$E^k(t) \left(W_m^{\mathfrak{sl}_n}(z|u_1, \dots, u_n) \otimes |\varnothing\rangle_H \right) = \frac{k!}{\mu^{-k}} T_k(t)\, W_m^{\mathfrak{sl}_n}(z|u_1, \dots, u_n) \otimes \exp\left(\frac{k}{n} \varphi_-(t) \right) |\varnothing\rangle_H \tag{1.13.27}$$

$$F^k(t) \left(W_m^{\mathfrak{sl}_n}(z|u_1, \dots, u_n) \otimes |\varnothing\rangle_H \right) = \frac{k!}{\mu^{k}} T_{n-k}(t)\, W_m^{\mathfrak{sl}_n}(z|u_1, \dots, u_n) \otimes \exp\left(-\frac{k}{n} \varphi_-(t) \right) |\varnothing\rangle_H \tag{1.13.28}$$

Moreover $\varphi_-(t)$ has only terms of positive degree in t. Hence we expressed action of $E^k[l]$ via $\tilde{T}_k[s]$ for $s \geqslant l$. Therefore we have proven that $W_m^{\mathfrak{sl}_n}(z|u_1, \dots, u_n) \otimes |\varnothing\rangle_H$ satisfies property of $W_m^{\mathfrak{gl}_n}(z|u_1, \dots, u_n)$.

The implication in opposite direction is analogous. ☐

Proposition 1.13.8. *There exists at most one vector $W_m^{\mathfrak{gl}_n}(z|u_1, \dots, u_n) \in \mathcal{F}_{u_1} \otimes \dots \otimes \mathcal{F}_{u_n}$.*

Proof. Analogous to proof of Proposition 1.13.2. The only difference is that we consider a different character of subalgebra $\mathfrak{n}$. ☐

We need to generalize notion of Whittaker vector for $\mathfrak{Diff}_q$ to compare it with $W_m^{\mathfrak{sl}_n}(z|u_1, \dots, u_n)$.

Definition 1.13.4. *For any $m \in \mathbb{Z}$, Whittaker vector $W_m(z|u_1, \dots, u_n) \in \mathcal{F}_{u_1} \otimes \dots \otimes \mathcal{F}_{u_n}$ is an eigenvector of operators $E_{a,b}$ for $mb \geqslant a \geqslant -(n-m)b$ and $b > 0$. More precisely,*

$$E_{-(n-m)k,k}\, W_m(z|u_1, \dots, u_n) = \frac{z^k}{q^{k/2} - q^{-k/2}} W_m(z|u_1, \dots, u_n) \tag{1.13.29}$$

$$E_{mk,k}\, W_m(z|u_1, \dots, u_n) = \frac{\left((-q^{\frac{1}{2}})^n u_1 \cdots u_n z\right)^k}{q^{-k/2} - q^{k/2}} W_m(z|u_1, \dots, u_n) \tag{1.13.30}$$

for $k > 0$.

$$E_{k_1,k_2} W_m(z) = 0 \tag{1.13.31}$$

for $(n-m)k_2 > k_1 > -mk_2$ and $k_2 > 0$. We require $W_m(z|u_1, \dots, u_n) = |\varnothing\rangle \otimes \dots \otimes |\varnothing\rangle + \cdots$ to fix normalization (by dots we mean lower vectors).

Recall that we have defined operator $\mathrm{I}_\tau \in \mathrm{End}(\mathcal{F}_u)$ by (1.3.23). By Proposition 1.3.8 the operator enjoys intertwiner property $\mathrm{I}_\tau \rho(E_{a,b}) \mathrm{I}_\tau^{-1} = \rho(E_{a-b,b})$. Denote $\mathrm{I}_{\tau,n} = \mathrm{I}_\tau \otimes \dots \otimes \mathrm{I}_\tau \in \mathrm{End}(\mathcal{F}_{u_1} \otimes \ldots \otimes \mathcal{F}_{u_n})$. Note that $I_{\tau,n}$ also enjoys intertwiner property $\mathrm{I}_\tau \rho_n(E_{a,b}) \mathrm{I}_\tau^{-1} = \rho_n(E_{a-b,b})$ (here ρ_n denotes the homomorphism of the representation $\rho_n \colon \mathfrak{Diff}_q \to \mathrm{End}(\mathcal{F}_{u_1} \otimes \ldots \otimes \mathcal{F}_{u_n})$).

Proposition 1.13.9. $W_m(z|u_1,\ldots,u_n) = \mathrm{I}_{\tau,n}^{n-m}\, W(z|u_1,\ldots,u_n)$.

Corollary 1.13.10. *There exists unique* $W_m(z|u_1,\ldots,u_n)$ *if* $u_i/u_j \neq q^k$.

Proposition 1.13.11. *There exists unique vector* $W_m^{\mathfrak{gl}_n}(z|u_1,\ldots,u_n)$. *Moreover,* $W_m^{\mathfrak{gl}_n}(z|u_1,\ldots,u_n) = W_m(z|u_1,\ldots,u_n)$.

Proof. We already know uniqueness of $W_m^{\mathfrak{gl}_n}(z|u_1,\ldots,u_n)$ and existence of $W_m(z|u_1,\ldots,u_n)$ from Proposition 1.13.8 and Corollary 1.13.10 correspondingly. So it is sufficient to show that $W_m(z|u_1,\ldots,u_n)$ satisfies properties of $W_m^{\mathfrak{gl}_n}(z|u_1,\ldots,u_n)$. Last assertion follows from formula (1.13.1) (also see (1.13.17)). □

Theorem 1.13.3. *There exists unique vector* $W_m^{\mathfrak{sl}_n}(z|u_1,\ldots,u_n)$. *Moreover,* $W_m^{\mathfrak{sl}_n}(z|u_1,\ldots,u_n) \otimes |\varnothing\rangle_H = W_m(z|u_1,\ldots,u_n) = \mathrm{I}_{\sigma,n}^{n-m}\, W(z|u_1,\ldots,u_n)$.

Proof. Follows from Lemma 1.13.7 and Proposition 1.13.11 □

Chapter 2

Twisted and non-twisted Virasoro

2.1 Introduction

The work is devoted to a probably new connection between deformed Virasoro algebra and quantum quantum affine algebra $\mathfrak{sl}_2$, denoted by $U_q(\widehat{\mathfrak{sl}}_2)$. More specifically, we use integrable representations $V(\Lambda_0)$ and $V(\Lambda_1)$ of $U_q(\widehat{\mathfrak{sl}}_2)$ on level 1, they possess realization in terms of Heisenberg algebra. Also there are vertex operators

$$\Phi(z)\colon V(\Lambda_i) \to V(\Lambda_{1-i}) \otimes V_z, \qquad \Psi(z)\colon V(\Lambda_i) \to V_z \otimes V(\Lambda_{1-i}). \tag{2.1.1}$$

for the tautological evaluation representation V_z of $U_q(\widehat{\mathfrak{sl}}_2)$. The main result is a realization of deformed Virasoro algebra in terms of these vertex operators see Theorem 2.4.1 and Theorem 2.4.2. Let us remark that deformed Virasoro algebra depend on parameters q_1, q_2, q_3 such that $q_1q_2q_3 = 1$. It turns out, that deformed Virasoro algebra is connected with $U_q(\widehat{\mathfrak{sl}}_2)$ for $q = q_3^{1/2}$.

To be more precise, in Theorem 2.4.2 we have a realization of twisted deformed Virasoro algebra. This algebra was defined in [Shi04, (37)–(38)] but its bosonization was unknown.

In Theorem 2.4.1 we have constructed realization of ordinary (non-twisted) deformed Virasoro algebra defined in [SKAO96]. A bosonization of this algebra is known since [SKAO96], but our bosonization is a different one. The bosonization from [SKAO96] can be also realized by the same formula as in Theorem 2.4.1 but using another vertex operators [DI97a] defined by (2.1.1) with respect to Drinfeld coproduct.

The existence of two realizations (our Theorems 2.4.1 and 2.4.2) is similar to the existence of two choices of twist in XXZ model, see e.g. [MN18, eq. 4.3].

Further development Deformed Virasoro algebra is a particular case of $W_{q_1,q_2}(\mathfrak{sl}_n)$ for $n = 2$. Twisted deformed Virasoro is a particular case of twisted W-algebras $W_{q_1,q_2}(\mathfrak{sl}_n, n_{tw})$ for $n = 2$ and $n_{tw} = 1$. Generally n_{tw} is a parameter of twist, i.e. for $n_{tw} = 0$ we obtain non-twisted W-algebra. We expect that one can construct a bosonization of $W_{q_1,q_2}(\mathfrak{sl}_n, n_{tw})$ algebra from the vertex operators of quantum $\widehat{\mathfrak{sl}}_n$ on the level 1 (see [Koy94]).

Also we expect that the tensor product of $W_{q_1,q_2}(\mathfrak{sl}_n, n_{tw})$ with Heisenberg algebra Heis are certain quotients of toroidal algebra $U_{q_1,q_2,q_3}(\ddot{\mathfrak{gl}}_1)$; for non-twisted case such a relation is known [FHS+10], [FFJ+11b], [Neg18]. Hence a representation of $W_{q_1,q_2}(\mathfrak{sl}_n, n_{tw}) \otimes \mathsf{Heis}$ becomes a representation of $U_{q_1,q_2,q_3}(\ddot{\mathfrak{gl}}_1)$ automatically. We expect that for $\gcd(n, n_{tw}) = 1$, the discussed above bosonized representation of $W_{q_1,q_2}(\mathfrak{sl}_n, n_{tw})$ will lead to Fock modules of $U_{q_1,q_2,q_3}(\ddot{\mathfrak{gl}}_1)$ with slope n_{tw}/n. For the case $q_3 = 1$ all this was done in Chapter 1. Let us remark $W_{q_1,q_2}(\mathfrak{sl}_n, n_{tw}) \otimes \mathsf{Heis}$ acts on an integrable level 1 representations of $U_q(\widehat{\mathfrak{gl}}_n) = U_q(\widehat{\mathfrak{sl}}_n) \otimes \mathsf{Heis}$ if this holds without the Heis factors.

One of our motivations for this project comes from [GN17]. It was conjectured in *loc. cit.* that there is an action (with certain properties) of $U_q\widehat{\mathfrak{gl}}_n$ on the Fock module of toroidal algebra $U_{q_1,q_2,q_3}(\ddot{\mathfrak{gl}}_1)$

with slope n'/n. As it was explained above, we also expect that $U_q\widehat{\mathfrak{gl}}_n$ acts on the Fock module of toroidal algebra $U_{q_1,q_2,q_3}(\ddot{\mathfrak{gl}}_1)$. So we hope that both actions exist and coincide.

Our methods. The main technical tool of this chapter is R-matrix relations (Theorems 2.3.1 and 2.3.2). One can find these relations without delta-function term in [JM95]. In *loc. cit.* the parameters of vertex operator are numbers, but, in in this thesis, the parameters are formal variables. Therefore our formulas are close to formulas in *loc. cit.*, but have a different meaning and probably are new.

Technically, we write down formulas (2.4.5) and (2.4.24) for the current of deformed Virasoro algebra $T(z)$ via vertex operators $\Phi(z)$ and $\Psi^*(z)$, and then we check relations of deformed Virasoro algebra using interchanging relations for vertex operators. Delta-function term on the RHS of deformed Virasoro relation appears from the delta-function term in the R-matrix relation.

Plan of the chapter. The chapter is organized as follows

In Section 2.2 we recall the bosonization of $U_q(\widehat{\mathfrak{sl}}_2)$ and its vertex operators following [JM95].

In Section 2.3 we study relations for the vertex operators: interchanging relations (in particular R-matrix relations), and 'special point relations'.

In Section 2.4 we construct realizations of twisted and non-twisted Virasoro algebra via vertex operators of $U_q(\widehat{\mathfrak{sl}}_2)$. A connection of obtained representations with Verma modules is studied.

2.2 Bosonization of $U_q(\widehat{\mathfrak{sl}}_2)$ and its vertex operators

In this section, we will recall the bosonization of the level 1 representations of $U_q(\widehat{\mathfrak{sl}}_2)$ and its vertex operators. All this can be found in [JM95, Chapters 5,6]. Our notation almost coincides with [JM95]; however, there are differences in the normalization of the vertex operators.

Fock modules. Algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is generated by $x_k^{\pm}$, a_l for $k \in \mathbb{Z}$, $l \in \mathbb{Z}_{\neq 0}$, $K^{\pm 1}$ and central elements $\gamma^{\pm 1/2}$. These elements are called *Drinfeld generators*. The relations are [JM95, (5.3)–(5.7)], although let us recall

$$[a_k, a_l] = \delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}, \tag{2.2.1}$$

here $[n] = (q^n - q^{-n})/(q - q^{-1})$.

Denote by Λ_0, Λ_1 the fundamental weights of $\widehat{\mathfrak{sl}}_2$ and by α root of $\mathfrak{sl}_2 \subset \widehat{\mathfrak{sl}}_2$. The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ admits two basic representations $V(\Lambda_0)$ and $V(\Lambda_1)$. As vector spaces

$$V(\Lambda_i) = \mathbb{C}[a_{-1}, a_{-2}, \dots] \otimes \left(\oplus_n \mathbb{C}e^{\Lambda_i + n\alpha}\right). \tag{2.2.2}$$

As representations of Heisenberg subalgebra a_k, these modules are infinite sums of Fock modules

$$V_j = \mathbb{C}[a_{-1}, a_{-2}, \dots] \otimes \mathbb{C}e^{\Lambda_i + \lfloor \frac{j}{2} \rfloor \alpha} \quad \text{for } i \equiv j \bmod 2. \tag{2.2.3}$$

Let us define operators $e^{\pm\alpha}$ and ∂ as follows

$$e^{\pm\alpha}\left(f \otimes e^{\beta}\right) = f \otimes e^{\beta \pm \alpha}, \qquad \partial\left(f \otimes e^{\beta}\right) = (\alpha, \beta) f \otimes e^{\beta}. \tag{2.2.4}$$

The action of other $U_q(\widehat{\mathfrak{sl}}_2)$ generators is given by

$$K = q^{\partial}, \quad \gamma = q, \tag{2.2.5}$$

$$X^{\pm}(z) = \exp\left(\pm\sum_{n=1}^{\infty}\frac{a_{-n}}{[n]}q^{\mp n/2}z^n\right)\exp\left(\mp\sum_{n=1}^{\infty}\frac{a_n}{[n]}q^{\mp n/2}z^{-n}\right)e^{\pm\alpha}z^{\pm\partial}, \tag{2.2.6}$$

here $X^{\pm}(z) = \sum x_k^{\pm} z^{-k-1}$. The obtained representations are irreducible highest weight representations with highest vectors $|\Lambda_i\rangle = 1 \otimes e^{\Lambda_i} \in V(\Lambda_i)$.

Vertex operators. Vertex operator of $U_q(\widehat{\mathfrak{sl}}_2)$ are certain formal power series of operators

$$\Phi_{\pm}^{(1-i,i)}(z)\colon V(\Lambda_i)\to V(\Lambda_{1-i}), \qquad \Psi_{\pm}^{(1-i,i)}(z)\colon V(\Lambda_i)\to V(\Lambda_{1-i}). \tag{2.2.7}$$

Below we will abbreviate $\Phi_\pm(z)=\Phi_\pm^{(1-i,i)}(z)$, if a statement holds for both $i=0,1$.

A conceptual definition of these operators via certain intertwiner relations is given in [JM95, Chapter 6]. For us it is more convenient to give an *ad hoc* definition

$$\Phi_-(z)=\exp\left(\sum_{n=1}^{\infty}\frac{a_{-n}}{[2n]}q^{7n/2}z^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{a_n}{[2n]}q^{-5n/2}z^{-n}\right)e^{\alpha/2}(-q^3z)^{\partial/2}, \tag{2.2.8}$$

$$\Psi_+(z)=\exp\left(-\sum_{n=1}^{\infty}\frac{a_{-n}}{[2n]}q^{n/2}z^n\right)\exp\left(\sum_{n=1}^{\infty}\frac{a_n}{[2n]}q^{-3n/2}z^{-n}\right)e^{-\alpha/2}(-qz)^{-\partial/2}. \tag{2.2.9}$$

Operator $\Phi_+(z)$, $\Psi_-(z)$ are determined by each of the formulas

$$\Phi_+(z)=[\Phi_-(z),x_0^-]_q, \qquad q^2z\Phi_+(z)=[\Phi_-(z),x_1^-]_{q^{-1}}, \tag{2.2.10}$$

$$\Psi_-(z)=[\Psi_+(z),x_0^+]_q, \qquad (q^2z)^{-1}\Psi_-(z)=[\Psi_+(z),x_{-1}^+]_{q^{-1}}. \tag{2.2.11}$$

here we use the following notation $[A,B]_p=AB-pBA$.

We will also need the dual operator $\Psi^*_\varepsilon(z)=\Psi_{-\varepsilon}(q^2z)$. Then

$$\Psi^*_-(z)=\exp\left(-\sum\frac{a_{-n}}{[2n]}q^{5n/2}z^n\right)\exp\left(\sum\frac{a_n}{[2n]}q^{-7n/2}z^{-n}\right)e^{-\alpha/2}(-q^3z)^{-\partial/2}. \tag{2.2.12}$$

Denote

$$\alpha_\phi(x)=\frac{(q^4x;q^4)_\infty}{(q^2x;q^4)_\infty}, \qquad \alpha_\psi(x)=\frac{(q^2x;q^4)_\infty}{(x;q^4)_\infty}, \qquad \beta(x)=\frac{(qx;q^4)_\infty}{(q^3x;q^4)_\infty}. \tag{2.2.13}$$

It is straightforward to check that

$$(-q^3z)^{-1/2}\alpha_\phi(w/z)\Phi_-(z)\Phi_-(w)=\ :\Phi_-(z)\Phi_-(w):, \tag{2.2.14}$$

$$(-qz)^{-1/2}\alpha_\psi(w/z)\Psi_+(z)\Psi_+(w)=\ :\Psi_+(z)\Psi_+(w):, \tag{2.2.15}$$

$$(-q^3z)^{1/2}\beta(w/z)\Phi_-(z)\Psi^*_-(w)=\ :\Phi_-(z)\Psi^*_-(w):, \tag{2.2.16}$$

$$(-q^3w)^{1/2}\beta(z/w)\Psi^*_-(w)\Phi_-(z)=\ :\Phi_-(z)\Psi^*_-(w):. \tag{2.2.17}$$

here $:\ldots:$ stands for the normal ordering in terms of Heisenberg algebra. Then

$$z^{-1/2}\alpha_\phi(w/z)\Phi_-(z)\Phi_-(w)=w^{-1/2}\alpha_\phi(z/w)\Phi_-(w)\Phi_-(z), \tag{2.2.18}$$

$$z^{-1/2}\alpha_\psi(w/z)\Psi^*_-(z)\Psi^*_-(w)=w^{-1/2}\alpha_\psi(z/w)\Psi^*_-(w)\Psi^*_-(z), \tag{2.2.19}$$

$$z^{1/2}\beta(w/z)\Phi_-(z)\Psi^*_-(w)=w^{1/2}\beta(z/w)\Psi^*_-(w)\Phi_-(z). \tag{2.2.20}$$

π-involution Recall, that $U_q(\widehat{\mathfrak{sl}}_2)$ is also generated by e_i, f_i, t_i for $i=0,1$. These generators are called *Chevalley generators*. The connection with Drinfeld generators is as follows

$$t_1=K, \qquad x_0^+=e_1, \qquad x_0^-=f_1, \tag{2.2.21}$$

$$t_0=\gamma K^{-1}, \qquad x_1^-=e_0t_1, \qquad x_{-1}^+=t_1^{-1}f_0. \tag{2.2.22}$$

Let us consider an exterior automorphism π of $U_q(\widehat{\mathfrak{sl}}_2)$ given by $\pi(e_i)=e_{1-i}$, $\pi(f_i)=f_{1-i}$. Then π acts on the Drinfeld generators as follows

$$\pi(K)=\gamma K^{-1}, \qquad \pi(x_0^+)=x_1^-K^{-1}, \qquad \pi(x_0^-)=Kx_{-1}^+, \tag{2.2.23}$$

$$\pi(x_1^-)=\gamma x_0^+K^{-1}, \qquad \pi(x_{-1}^+)=\gamma^{-1}Kx_0^-. \tag{2.2.24}$$

Proposition 2.2.1. *There exist an involution $\tilde{\pi}$ interchanging $V(\Lambda_0)$ and $V(\Lambda_1)$, such that $\tilde{\pi}X\tilde{\pi} = \pi(X)$ for any $X \in U_q(\widehat{\mathfrak{sl}}_2)$.*

Proof. $V(\Lambda_0)$ and $V(\Lambda_1)$ are irreducible highest weights representations and π preserves triangular decomposition. To finish the proof we notice that action of π interchange the highest weights of the representations. □

To determine $\tilde{\pi}$ uniquely we require $\tilde{\pi}\left(|\Lambda_i\rangle\right) = |\Lambda_{1-i}\rangle$.

Proposition 2.2.2. *Conjugation by involution $\tilde{\pi}$ is expressed as follows*

$$\tilde{\pi}\left(\Phi_+^{(1-i,i)}(z)\right)\tilde{\pi} =(-q^3)^{\frac{1}{2}-i}z^{-\frac{1}{2}}\Phi_-^{(i,1-i)}(z), \qquad \tilde{\pi}\left(\Phi_-^{(1-i,i)}(z)\right)\tilde{\pi} =(-q^3)^{\frac{1}{2}-i}z^{\frac{1}{2}}\Phi_+^{(i,1-i)}(z), \tag{2.2.25}$$

$$\tilde{\pi}\left(\Psi_+^{(1-i,i)}(z)\right)\tilde{\pi} =(-q)^{3\left(\frac{1}{2}-i\right)}z^{-\frac{1}{2}}\Psi_-^{(i,1-i)}(z), \quad \tilde{\pi}\left(\Psi_-^{(1-i,i)}(z)\right)\tilde{\pi} =(-q)^{3\left(\frac{1}{2}-i\right)}z^{\frac{1}{2}}\Psi_+^{(i,1-i)}(z). \tag{2.2.26}$$

Sketch of a proof. One can prove the formulas up to a constant via the intertwining properties [JM95, Chapter 6]

$$\tilde{\pi}\left(\Phi_+^{(1-i,i)}(z)\right)\tilde{\pi} =c_1^{(i)}\Phi_-^{(i,1-i)}(z), \qquad \tilde{\pi}\left(\Phi_-^{(1-i,i)}(z)\right)\tilde{\pi} =c_1^{(i)}z\Phi_+^{(i,1-i)}(z), \tag{2.2.27}$$

$$\tilde{\pi}\left(\Psi_+^{(1-i,i)}(z)\right)\tilde{\pi} =c_2^{(i)}\Psi_-^{(i,1-i)}(z), \qquad \tilde{\pi}\left(\Psi_-^{(1-i,i)}(z)\right)\tilde{\pi} =c_2^{(i)}z\Psi_+^{(i,1-i)}(z), \tag{2.2.28}$$

here $c_1^{(i)}$ and $c_2^{(i)}$ are some z-dependent scalars. Then one can find the constants by comparison with the normalization [JM95, eq. (6.4), (6.5)]. □

Corollary 2.2.3. *The following relations hold*

$$z^{-1/2}\alpha_\phi(w/z)\Phi_+(z)\Phi_+(w) =w^{-1/2}\alpha_\phi(z/w)\Phi_+(w)\Phi_+(z), \tag{2.2.29}$$

$$z^{-1/2}\alpha_\psi(w/z)\Psi_+^*(z)\Psi_+^*(w) =w^{-1/2}\alpha_\psi(z/w)\Psi_+^*(w)\Psi_+^*(z), \tag{2.2.30}$$

$$z^{1/2}\beta(w/z)\Phi_+(z)\Psi_+^*(w) =w^{1/2}\beta(z/w)\Psi_+^*(w)\Phi_+(z). \tag{2.2.31}$$

Proof. These relations are obtained from (2.2.18)–(2.2.20) after conjugation by $\tilde{\pi}$. □

2.3 Vertex operators relations revisited

The main results of this section are R-matrix relations (Theorems 2.3.1 and 2.3.2). One can find these relations without delta-function term in [JM95]. In *loc. cit.* parameters of vertex operator are numbers, but in this thesis, the parameters are formal variables. Although formulas below are close to formulas in *loc. cit.*, they have a different meaning and can be considered as a new result.

2.3.1 R-matrix relations

R-matrix is an operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Let v_+, v_- be a basis of each $\mathbb{C}^2$. The matrix of this operator with respect to the basis $v_+ \otimes v_+$, $v_+ \otimes v_-$, $v_- \otimes v_+$, $v_- \otimes v_-$ looks as follows

$$R(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(1-x)q}{1-q^2x} & \frac{1-q^2}{1-q^2x} & 0 \\ 0 & \frac{(1-q^2)x}{1-q^2x} & \frac{(1-x)q}{1-q^2x} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.3.1}$$

This R-matrix is an important object in the representation theory of $U_q(\widehat{\mathfrak{sl}}_2)$, though in this chapter we will not use any information on R-matrix apart from (2.3.1). Below we will see that R-matrix encodes certain interchanging relations for vertex operators.

Interchanging relation on Φ-vertex operators

Denote by $\delta(x,y)=\sum_{k+l=-1}x^k y^l$.

Proposition 2.3.1. *The following relations hold*

$$z^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_-(z)\Phi_+(w)-w^{-\frac{1}{2}}\alpha_\phi(z/w)\left(\frac{q(1-z/w)}{q^2-z/w}\Phi_+(w)\Phi_-(z)+\frac{(q^2-1)z/w}{q^2-z/w}\Phi_-(w)\Phi_+(z)\right)$$
$$=(-1)^\partial(-q^3)^{\frac{1}{2}}q^{-2}\delta(z,q^2w),\quad(2.3.2)$$

$$z^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_+(z)\Phi_-(w)-w^{-\frac{1}{2}}\alpha_\psi(z/w)\left(\frac{q(1-z/w)}{q^2-z/w}\Phi_-(w)\Phi_+(z)+\frac{q^2-1}{q^2-z/w}\Phi_+(w)\Phi_-(z)\right)$$
$$=-(-1)^\partial(-q^3)^{\frac{1}{2}}q^{-3}\delta(z,q^2w).\quad(2.3.3)$$

Proof. Using (2.2.10), we obtain

$$[\Phi_-(z)\Phi_-(w),x_0^-]_{q^2}=\Phi_-(z)[\Phi_-(w),x_0^-]_q+q[\Phi_-(z),x_0^-]_q\Phi_-(w)$$
$$=\Phi_-(z)\Phi_+(w)+q\Phi_+(z)\Phi_-(w),\quad(2.3.4)$$
$$[\Phi_-(z)\Phi_-(w),x_1^-]_{q^{-2}}=\Phi_-(z)[\Phi_-(w),x_1^-]_{q^{-1}}+q^{-1}[\Phi_-(z),x_1^-]_{q^{-1}}\Phi_-(w)$$
$$=q^2w\Phi_-(z)\Phi_+(w)+qz\Phi_+(z)\Phi_-(w).\quad(2.3.5)$$

Solving the system of two linear equations, one can find

$$\Phi_-(z)\Phi_+(w)=\frac{z[\Phi_-(z)\Phi_-(w),x_0^-]_{q^2}-[\Phi_-(z)\Phi_-(w),x_1^-]_{q^{-2}}}{z(1-q^2w/z)},\quad(2.3.6)$$
$$\Phi_+(z)\Phi_-(w)=\frac{-qw[\Phi_-(z)\Phi_-(w),x_0^-]_{q^2}+q^{-1}[\Phi_-(z)\Phi_-(w),x_1^-]_{q^{-2}}}{z(1-q^2w/z)}.\quad(2.3.7)$$

Using (2.2.14), we see that

$$(-q^3z)^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_-(z)\Phi_+(w)=\frac{z[:\Phi_-(z)\Phi_-(w):,x_0^-]_{q^2}-[:\Phi_-(z)\Phi_-(w):,x_1^-]_{q^{-2}}}{z(1-q^2w/z)},\quad(2.3.8)$$
$$(-q^3z)^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_+(z)\Phi_-(w)=\frac{-qw[:\Phi_-(z)\Phi_-(w):,x_0^-]_{q^2}+q^{-1}[:\Phi_-(z)\Phi_-(w):,x_1^-]_{q^{-2}}}{z(1-q^2w/z)}.\quad(2.3.9)$$

Then

$$z^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_-(z)\Phi_+(w)-w^{-1/2}\alpha_\phi(z/w)\left(\frac{q(1-z/w)}{q^2-z/w}\Phi_+(w)\Phi_-(z)+\frac{(q^2-1)z/w}{q^2-z/w}\Phi_-(w)\Phi_+(z)\right)$$
$$=(-q^3)^{\frac{1}{2}}\Big(q^2w[:\Phi_-(q^2w)\Phi_-(w):,x_0^-]_{q^2}-[:\Phi_-(q^2w)\Phi_-(w):,x_1^-]_{q^{-2}}\Big)\delta(z,q^2w),\quad(2.3.10)$$

$$z^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_+(z)\Phi_-(w)-w^{-\frac{1}{2}}\alpha_\phi(z/w)\left(\frac{q(1-z/w)}{q^2-z/w}\Phi_-(w)\Phi_+(z)+\frac{q^2-1}{q^2-z/w}\Phi_+(w)\Phi_-(z)\right)$$
$$=(-q^3)^{\frac{1}{2}}\Big(-qw[:\Phi_-(q^2w)\Phi_-(w):,x_0^-]_{q^2}+q^{-1}[:\Phi_-(q^2w)\Phi_-(w):,x_1^-]_{q^{-2}}\Big)\delta(z,q^2w).\quad(2.3.11)$$

Lemma 2.3.2. *The following relation holds*

$$q^2w[:\Phi_-(q^2w)\Phi_-(w):,x_0^-]_{q^2}-[:\Phi_-(q^2w)\Phi_-(w):,x_1^-]_{q^{-2}}=(-1)^\partial q^{-2}.\quad(2.3.12)$$

Proof. We will proof the lemma assuming w to be a number, not a formal variable; the formal variable version follows. Let us consider two contours of integration $C_+ = \{y \mid |y| = R_+ \gg |w|\}$, $C_- = \{y \mid |y| = R_- \ll |w|\}$

Denote $\Omega(w) = :\Phi_-(q^2w)\Phi_-(w):$. Note that

$$[\Omega(w), x_0^-]_{q^2} = \int_{C_-} \Omega(w)X^-(y)dy - q^2 \int_{C_+} X^-(y)\Omega(w)dy, \tag{2.3.13}$$

$$[\Omega(w), x_1^-]_{q^{-2}} = \int_{C_-} y\Omega(w)X^-(y)dy - q^{-2} \int_{C_+} yX^-(y)\Omega(w)dy. \tag{2.3.14}$$

Hence

$$\begin{aligned} q^2w[\Omega(w), x_0^-]_{q^2} - [\Omega(w), x_1^-]_{q^{-2}} &= \int_{C_-} \big(q^2w - y\big)\Omega(w)X^-(y)dy - \int_{C_+} \big(q^4w - q^{-2}y\big)X^-(y)\Omega(w)dy \\ = \int_{C_-} \frac{\big(q^2w-y\big)}{q^2(q^4w-y)(q^2w-y)} :\Omega(w)X^-(y): dy &- \int_{C_+} \frac{q^4w - q^{-2}y}{(y-q^4w)(y-q^6w)} :X^-(y)\Omega(w): dy \\ = \int_{C_+} \frac{q^{-2}}{y-q^4w} :X^-(y)\Omega(w): dy &- \int_{C_-} \frac{q^{-2}}{y-q^4w} :\Omega(w)X^-(y): dy \\ &= \mathrm{res}_{y=q^4w} \frac{q^{-2}}{y-q^4w} :X^-(y)\Omega(w): dy. = (-1)^\partial q^{-2}. \end{aligned}$$

Here we used $:X^-(q^4w)\Omega(w):= (-1)^\partial$. ☐

To finish the proof of Proposition 2.3.1 we apply Lemma 2.3.2 to (2.3.10) and (2.3.11). ☐

Matrix notation Denote $\Phi(z) = \Phi_+(z) \otimes v_+ + \Phi_-(z) \otimes v_- \in \mathrm{Hom}(V(\Lambda_i), V(\Lambda_{1-i})) \otimes \mathbb{C}^2$. Denote products

$$\Phi^{(1)}(z)\Phi^{(2)}(w) = \sum_{\epsilon_1,\epsilon_2=\pm} \Phi_{\epsilon_1}(z)\Phi_{\epsilon_2}(w) \otimes v_{\epsilon_1} \otimes v_{\epsilon_2}, \quad \Phi^{(2)}(w)\Phi^{(1)}(z) = \sum_{\epsilon_1,\epsilon_2=\pm} \Phi_{\epsilon_2}(w)\Phi_{\epsilon_1}(z) \otimes v_{\epsilon_1} \otimes v_{\epsilon_2}.$$

Finally denote by $R^{-1}(z/w)\Phi^{(2)}(w)\Phi^{(1)}(z)$ the result of the action of $R^{-1}(z/w)$ on the $\mathbb{C}^2 \otimes \mathbb{C}^2$ tensor multiple of $\mathrm{Hom}(V(\Lambda_i), V(\Lambda_i)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

Theorem 2.3.1. *The following relation holds*

$$\begin{aligned} &z^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi^{(1)}(z)\Phi^{(2)}(w) = \\ &w^{-\frac{1}{2}}\alpha_\phi(z/w)R^{-1}(z/w)\Phi^{(2)}(w)\Phi^{(1)}(z) + (-1)^\partial(-q)^{\frac{1}{2}}(q^{-1}v_- \otimes v_+ - q^{-2}v_+ \otimes v_-)\delta(z, q^2w). \end{aligned} \tag{2.3.15}$$

Proof. The theorem is just a reformulation of (2.2.18), (2.2.29) and Proposition 2.3.1. ☐

Interchanging relation for Ψ-vertex operators

Proposition 2.3.3. *The following relations hold*

$$\begin{aligned} z^{-1/2}\alpha_\psi(w/z)\Psi_+(z)\Psi_-(w) - w^{-1/2}\alpha_\psi(z/w)\left(\frac{q(1-z/w)}{1-q^2z/w}\Psi_-(w)\Psi_+(z) + \frac{1-q^2}{1-q^2z/w}\Psi_+(w)\Psi_-(z)\right) \\ = (-1)^\partial(-q)^{\frac{1}{2}}q^2\delta(q^2z, w), \end{aligned} \tag{2.3.16}$$

$$\begin{aligned} z^{-1/2}\alpha_\psi(w/z)\Psi_-(z)\Psi_+(w) - w^{-1/2}\alpha_\psi(z/w)\left(\frac{q(1-\frac{z}{w})}{1-q^2\frac{z}{w}}\Psi_+(w)\Psi_-(z) + \frac{(1-q^2)\frac{z}{w}}{1-q^2\frac{z}{w}}\Psi_-(w)\Psi_+(z)\right) \\ = -(-1)^\partial(-q)^{\frac{1}{2}}q\delta(q^2z, w). \end{aligned} \tag{2.3.17}$$

Proof. Using (2.2.11) we obtain

$$[\Psi_+(z)\Psi_+(w), x_0^+]_{q^2} = \Psi_+(z)[\Psi_+(w), x_0^+]_q + q[\Psi_+(z), x_0^+]_q\Psi_+(w) = \Psi_+(z)\Psi_-(w) + q\Psi_-(z)\Psi_+(w),$$

$$\begin{aligned}[\Psi_+(z)\Psi_+(w), x_{-1}^+]_{q^{-2}} &= \Psi_+(z)[\Psi_+(w), x_{-1}^+]_{q^{-1}} + q^{-1}[\Psi_+(z), x_{-1}^+]_{q^{-1}}\Psi_+(w) \\ &= (q^2w)^{-1}\Psi_+(z)\Psi_-(w) + (q^3z)^{-1}\Psi_-(z)\Psi_+(w).\end{aligned}$$

Solving the system of linear equations, we obtain

$$\Psi_+(z)\Psi_-(w) = \frac{w/z\left(-[\Psi_+(z)\Psi_+(w), x_0^+]_{q^2} + q^4z[:\Psi_+(z)\Psi_+(w):, x_{-1}^+]_{q^{-2}}\right)}{q^2 - w/z}, \tag{2.3.18}$$

$$\Psi_-(z)\Psi_+(w) = \frac{q[\Psi_+(z)\Psi_+(w), x_0^+]_{q^2} - q^3w[:\Psi_+(z)\Psi_+(w), x_{-1}^+]_{q^{-2}}}{q^2 - w/z}. \tag{2.3.19}$$

Using (2.2.15), we see that

$$(-qz)^{-1/2}\alpha_\psi(w/z)\Psi_+(z)\Psi_-(w) = \frac{w\left(-[:\Psi_+(z)\Psi_+(w):, x_0^+]_{q^2} + q^4z[:\Psi_+(z)\Psi_+(w):, x_{-1}^+]_{q^{-2}}\right)}{q^2z\left(1 - \frac{w}{q^2z}\right)}, \tag{2.3.20}$$

$$(-qz)^{-1/2}\alpha_\psi(w/z)\Psi_-(z)\Psi_+(w) = \frac{q[:\Psi_+(z)\Psi_+(w):, x_0^+]_{q^2} - q^3w[:\Psi_+(z)\Psi_+(w):, x_{-1}^+]_{q^{-2}}}{q^2\left(1 - \frac{w}{q^2z}\right)}. \tag{2.3.21}$$

Then

$$\begin{aligned} z^{-\frac{1}{2}}\alpha_\psi(w/z)\Psi_+(z)\Psi_-(w) - w^{-\frac{1}{2}}\alpha_\psi(z/w)\left(\frac{1-q^2}{1-q^2z/w}\Psi_+(w)\Psi_-(z) + \frac{q(1-z/w)}{1-q^2z/w}\Psi_-(w)\Psi_+(z)\right) \\ = (-q)^{\frac{1}{2}}q^2z\Big(-[:\Psi_+(z)\Psi_+(q^2z):, x_0^+]_{q^2} + q^4z[:\Psi_+(z)\Psi_+(q^2z):, x_{-1}^+]_{q^{-2}}\Big)\ \delta(q^2z, w). \end{aligned} \tag{2.3.22}$$

$$\begin{aligned} z^{-\frac{1}{2}}\alpha_\psi(w/z)\Psi_-(z)\Psi_+(w) - w^{-\frac{1}{2}}\alpha_\psi(z/w)\left(\frac{q(1-z/w)}{1-q^2z/w}\Psi_+(w)\Psi_-(z) + \frac{(1-q^2)z/w}{1-q^2z/w}\Psi_-(w)\Psi_+(z)\right) \\ = (-q)^{\frac{1}{2}}qz\Big([:\Psi_+(z)\Psi_+(q^2z):, x_0^+]_{q^2} - q^4z[:\Psi_+(z)\Psi_+(q^2z):, x_{-1}^+]_{q^{-2}}\Big)\delta(q^2z, w). \end{aligned} \tag{2.3.23}$$

Lemma 2.3.4. *The following relation holds*

$$[:\Psi_+(z)\Psi_+(q^2z):, x_0^+]_{q^2} - q^4z[:\Psi_+(z)\Psi_+(q^2z):, x_{-1}^+]_{q^{-2}} = -(-1)^\partial z^{-1}. \tag{2.3.24}$$

Proof. Denote by

$$\Upsilon(z) = :\Psi_+(z)\Psi_+(q^2z):. \tag{2.3.25}$$

Note that

$$[\Upsilon(z), x_0^+]_{q^2} = \int_{C_-}\Upsilon(z)X^+(y)dy - q^2\int_{C_+}X^+(y)\Upsilon(z)dy, \tag{2.3.26}$$

$$[\Upsilon(z), x_{-1}^+]_{q^{-2}} = \int_{C_-}y^{-1}\Upsilon(z)X^+(y)dy - q^{-2}\int_{C_+}y^{-1}X^+(y)\Upsilon(z)dy. \tag{2.3.27}$$

Hence

$$
\begin{aligned}
[\Upsilon(z), x_0^+]_{q^2} - q^4 z[\Upsilon(z), x_{-1}^+]_{q^{-2}} &= \int_{C_-} (1-q^4 z/y)\Upsilon(z) X^+(y)dy - q^2 \int_{C_+} (1-z/y) X^+(y)\Upsilon(z)dy \\
&= \int_{C_-} \frac{q^2(1-q^4z/y)}{(q^2z-y)(q^4z-y)} :\Upsilon(z)X^+(y): dy - q^2 \int_{C_+} \frac{(1-z/y)}{(y-z)(y-q^2z)} :X^+(y)\Upsilon(z): dy \\
&= -\int_{C_-} \frac{q^2}{y(q^2z-y)} :\Upsilon(z)X^+(y): dy + \int_{C_+} \frac{q^2}{y(q^2z-y)} :X^+(y)\Upsilon(z): dy \\
&= \mathrm{res}_{y=q^2z} \frac{q^2}{y(q^2z-y)} :X^+(y)\Upsilon(z): dy = -(-1)^{\partial} z^{-1}.
\end{aligned}
$$

Here we used $:X^+(q^2z)\Upsilon(z): = (-1)^{\partial}$. □

□

In terms of the operators Ψ^*, Proposition 2.3.3 can be rewritten as follows.

Corollary 2.3.5. *The following relation holds*

$$
\begin{aligned}
z^{-1/2}\alpha_\psi(w/z)\Psi^*_-(z)\Psi^*_+(w) - w^{-1/2}\alpha_\psi(z/w)\left(\frac{q(1-z/w)}{1-q^2z/w}\Psi^*_+(w)\Psi^*_-(z) + \frac{1-q^2}{1-q^2z/w}\Psi^*_-(w)\Psi^*_+(z)\right) \\
= (-1)^{\partial}(-q)^{\frac{1}{2}} q\delta(q^2z,w), \quad (2.3.28)
\end{aligned}
$$

$$
\begin{aligned}
z^{-1/2}\alpha_\psi(w/z)\Psi^*_+(z)\Psi^*_-(w) - w^{-1/2}\alpha_\psi(z/w)\left(\frac{q(1-\frac{z}{w})}{1-q^2\frac{z}{w}}\Psi^*_-(w)\Psi^*_+(z) + \frac{(1-q^2)\frac{z}{w}}{1-q^2\frac{z}{w}}\Psi^*_+(w)\Psi^*_-(z)\right) \\
= -(-1)^{\partial}(-q)^{\frac{1}{2}}\delta(q^2z,w). \quad (2.3.29)
\end{aligned}
$$

Matrix notation Denote $\Psi^*(z) = \Psi^*_+(z)\otimes v^*_+ + \Psi^*_-(z)\otimes v^*_- \in \mathrm{Hom}(V(\Lambda_i), V(\Lambda_{1-i}))\otimes \left(\mathbb{C}^2\right)^*$. Let us emphasise that $\left(\mathbb{C}^2\right)^*$ is the dual space to $\mathbb{C}^2$, considered in the definition of $\Phi(z)$.

Denote products

$$
\Psi^{*,(1)}(z)\Psi^{*,(2)}(w) = \sum_{\epsilon_1,\epsilon_2=\pm 1} \Psi^*_{\epsilon_1}(z)\Psi^*_{\epsilon_2}(w)\otimes v^*_{\epsilon_1}\otimes v^*_{\epsilon_2}, \quad (2.3.30)
$$

$$
\Psi^{*,(2)}(w)\Psi^{*,(1)}(z) = \sum_{\epsilon_1,\epsilon_2=\pm 1} \Psi^*_{\epsilon_2}(w)\Psi^*_{\epsilon_1}(z)\otimes v^*_{\epsilon_1}\otimes v^*_{\epsilon_2}. \quad (2.3.31)
$$

Finally denote by $\Psi^{*,(2)}(w)\Psi^{*,(1)}(z)R(z/w)$ the result of the dual action of $R(z/w)$ on the $\left(\mathbb{C}^2\otimes\mathbb{C}^2\right)^*$ tensor multiple of $\mathrm{Hom}(V(\Lambda_i), V(\Lambda_i))\otimes\left(\mathbb{C}^2\otimes\mathbb{C}^2\right)^*$. In other words, we multiply the operator-valued *row vector* on the matrix.

Theorem 2.3.2. *The following relation holds*

$$
\begin{aligned}
& z^{-\frac{1}{2}}\alpha_\psi(w/z)\Psi^{*,(1)}(z)\Psi^{*,(2)}(w) \\
&= w^{-\frac{1}{2}}\alpha_\psi(z/w)\Psi^{,*(2)}(w)\Psi^{*,(1)}(z)R(z/w) + (-1)^{\partial}(-q)^{\frac{1}{2}}(qv^*_-\otimes v^*_+ - v^*_+\otimes v^*_-)\delta(q^2z,w). \quad (2.3.32)
\end{aligned}
$$

Proof. The theorem is just a reformulation of (2.2.19), (2.2.30), and Corollary 2.3.5. □

2.3.2 Special point relation

Special point for Φ

Proposition 2.3.6. *We have the following identity*

$$(-qz)^{-1/2}\alpha_{\phi}(w/z)\left(q\Phi_-(z)\Phi_+(w)-\Phi_+(z)\Phi_-(w)\right)\Big|_{w=q^2z}=\frac{(-1)^{\partial}}{zq^2(1-q^2)}. \tag{2.3.33}$$

Remark 2.3.1. Note, that *a priori* the LHS of (2.3.33) is not well defined since a coefficient of any power of z is an infinite sum of operators. So we have to prove that the result of the substitution exists as well as to find the result. Also note, that we substitute $w\mapsto q^2z$ to the whole expression, not to the individual multiples; the result of the substitution to the individual multiples does not have to exist.

Proof. Substituting $w\mapsto q^2z$ to (2.3.8) and (2.3.9) we obtain

$$(-q^3z)^{-1/2}\alpha_{\phi}(w/z)\left(q\Phi_-(z)\Phi_+(w)-\Phi_+(z)\Phi_-(w)\right)\Big|_{w=q^2z}= \\ \frac{q^2z[:\Phi_-(z)\Phi_-(q^2z):,x_0^-]_{q^2}-[:\Phi_-(z)\Phi_-(q^2z):,x_1^-]_{q^{-2}}}{qz(1-q^2)}. \tag{2.3.34}$$

To finish the proof we apply Lemma 2.3.2. □

Special point for Ψ

Proposition 2.3.7. *We have the following identity*

$$(-qz)^{-1/2}\alpha_{\psi}(w/z)\left(\Psi_-(z)\Psi_+(w)-q\Psi_+(z)\Psi_-(w)\right)\Big|_{z=q^2w}=\frac{qw^{-1}}{1-q^2}(-1)^{\partial}. \tag{2.3.35}$$

Proof. Let us substitute $z\mapsto q^2w$ to (2.3.20) and (2.3.21)

$$(-qz)^{-1/2}\alpha_{\psi}(w/z)\left(\Psi_-(z)\Psi_+(w)-q\Psi_+(z)\Psi_-(w)\right)\Big|_{z=q^2w}= \\ \frac{q[:\Psi_+(q^2w)\Psi_+(w):,x_0^+]_{q^2}-q^5w[:\Psi_+(q^2w)\Psi_+(w):,x_{-1}^+]_{q^{-2}}}{q^2-1}. \tag{2.3.36}$$

To finish the proof we applying Lemma 2.3.4. □

Corollary 2.3.8. *For any $q_1\in\mathbb{C}\backslash\{0\}$ we have the following identity*

$$(-q_1/z)^{1/2}\alpha_{\psi}(w/z)\left(\Psi_+^*(qq_1z)\Psi_-^*(qq_1w)-q\Psi_-^*(qq_1z)\Psi_+^*(qq_1w)\right)\Big|_{z=q^2w}=-\frac{w^{-1}}{1-q^2}(-1)^{\partial}. \tag{2.3.37}$$

2.3.3 Interchanging relation on Φ and Ψ

Proposition 2.3.9. *The following relation holds*

$$z^{\frac12}\beta(w/z)\Phi_{\epsilon_1}(z)\Psi_{\epsilon_2}^*(w)=w^{\frac12}\beta(z/w)\Psi_{\epsilon_2}^*(w)\Phi_{\epsilon_1}(z). \tag{2.3.38}$$

Proof. We have already seen the cases $\epsilon_1=\epsilon_2=\pm$, see (2.2.20) and (2.2.31). To prove the remaining cases, let us use a relation from [JM95, Section 6.3] and relation [JM95, (6.12)]

$$\Psi_-^*(z)x_0^- - x_0^-\Psi_-^*(z)=0, \qquad \Phi_-(z)x_0^+ - x_0^+\Phi_-(z)=0. \tag{2.3.39}$$

To be combined with (2.2.10) and (2.2.11), the relations yield

$$[\Phi_-(z)\Psi^*_-(w), x_0^-]_q = [\Phi_-(z), x_0^-]_q \Psi^*_-(w) = \Phi_+(z)\Psi^*_-(w), \tag{2.3.40}$$
$$[\Phi_-(z)\Psi^*_-(w), x_0^+]_q = \Phi_-(z)[\Psi^*_-(w), x_0^+]_q = \Phi_-(z)\Psi^*_+(w). \tag{2.3.41}$$

Considering q-commutator of (2.2.20) with x_0^- and x_0^+, we obtain the cases $\epsilon_1 = +$, $\epsilon_2 = -$ and $\epsilon_1 = -$, $\epsilon_2 = +$ correspondingly. □

2.4 Realization of (Twisted) Deformed Virasoro algebra

In this section, we will consider two algebras: *deformed Virasoro algebra* and *twisted deformed Virasoro algebra*. Deformed Virasoro algebra is extensively studied. Twisted Virasoro was defined in [Shi04], though this algebra is considerably less famous.

The algebras depend on two parameters q_1, q_2. It is also convenient to consider q_3 such that $q_1q_2q_3 = 1$. In this section we study a connection between the algebras and $U_q(\widehat{\mathfrak{sl}}_2)$ for $q^2 = q_3$.

To define (twisted) deformed Virasoro algebra, we need the following notation

$$\sum_{l=0}^{\infty} f_l x^l = f(x) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)}{1+q_3^{-n}} x^n\right). \tag{2.4.1}$$

Note that

$$f(x) = \frac{1}{1-x}\beta\left(q_1 q x\right)\beta\left(q_1^{-1}q^{-1}x\right). \tag{2.4.2}$$

2.4.1 Deformed Virasoro algebra

Definition 2.4.1. *Deformed Virasoro algebra* Vir_{q_1,q_2} *is generated by* T_n *for* $n \in \mathbb{Z}$. *The defining relation is*

$$\sum_{l=0}^{\infty} f_l T_{n-l}T_{m+l} - \sum_{l=0}^{\infty} f_l T_{n-l}T_{m+l} = -\frac{(1-q_1)(1-q_2)}{1-q_3^{-1}}\left(q_3^{-n} - q_3^{n}\right)\delta_{n+m,0}. \tag{2.4.3}$$

Denote $T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n}$, $\delta(x) = \sum_{k\in\mathbb{Z}} x^k$. *Relation* (2.4.3) *is equivalent to*

$$f(w/z)T(z)T(w) - f(z/w)T(w)T(z) = -\frac{(1-q_1)(1-q_2)}{1-q_3^{-1}}\left(\delta\left(\frac{w}{q_3 z}\right) - \delta\left(\frac{q_3 w}{z}\right)\right). \tag{2.4.4}$$

Representation Recall that V_j were defined by (2.2.3).

Theorem 2.4.1. *The formula below determines an action of* Vir_{q_1,q_2} *on* V_j *for all* $j \in \mathbb{Z}$.

$$T(z) = z^{1/2}\frac{q^{3/2}(q_1^{1/2} - q_1^{-1/2})}{\beta(q/q_1)}\Big(u\Psi^*_+(qq_1z)\Phi_+(z) + u^{-1}\Psi^*_-(qq_1z)\Phi_-(z)\Big). \tag{2.4.5}$$

Denote the obtained representation by $\mathcal{F}_u^{[j]}$.

Remark 2.4.1. A bosonization of deformed Virasoro algebra is known since [SKAO96], but our bosonization is a different one. In both cases current $T(z)$ is presented as a sum of two summands. Surprisingly, the first summands in both cases are 'the same normally ordered exponent of Heisenberg a_k'; however, the second ones are different. In [SKAO96] the second summand is also an exponent of the same Heisenberg, but this is not true for our bosonization. Note that $\Psi^*_-(qq_1z)\Phi_-(z)$ is an exponent of Heisenberg $\pi(a_k)$, but not of a_k.

Proof. The proof is basically a verification of (2.4.4). Let us rewrite (2.4.5) in the matrix form

$$T(z)=\frac{q^{3/2}(q_1^{1/2}-q_1^{-1/2})}{\beta(q/q_1)}\Psi^*(qq_1z)\varepsilon_z\Phi(z),\ \text{for}\ \varepsilon_z=z^{1/2}\begin{pmatrix}u&0\\0&u^{-1}\end{pmatrix}. \tag{2.4.6}$$

Using (2.4.2), Proposition 2.3.9 and (2.2.13) we obtain

$$\begin{aligned}
&f(w/z)\left(\Psi^*(qq_1z)\varepsilon_z\Phi(z)\right)\left(\Psi^*(qq_1w)\varepsilon_w\Phi(w)\right)\\
&=\frac{1}{1-w/z}\beta\left(\frac{q_1qw}{z}\right)\beta\left(\frac{w}{q_1qz}\right)\left(\Psi^*(qq_1z)\varepsilon_z\Phi(z)\right)\left(\Psi^*(qq_1w)\varepsilon_w\Phi(w)\right)\\
&=\frac{1}{1-w/z}\left(\frac{qq_1w}{z}\right)^{\frac12}\beta\left(\frac{z}{q_1qw}\right)\beta\left(\frac{w}{q_1qz}\right)\left(\Psi^{*,(1)}(qq_1z)\Psi^{*,(2)}(qq_1w)\right)\varepsilon_z\otimes\varepsilon_w\left(\Phi^{(1)}(z)\Phi^{(2)}(w)\right)\\
&=\beta\left(\frac{z}{q_1qw}\right)\beta\left(\frac{w}{q_1qz}\right)(qq_1zw)^{\frac12}\\
&\times\left(z^{-\frac12}\alpha_\psi\left(\frac{w}{z}\right)\Psi^{*,(1)}(qq_1z)\Psi^{*,(2)}(qq_1w)\right)\varepsilon_z\otimes\varepsilon_w\left(z^{-\frac12}\alpha_\phi\left(\frac{w}{z}\right)\Phi^{(1)}(z)\Phi^{(2)}(w)\right).
\end{aligned}\tag{2.4.7}$$

To continue the calculation, we apply Theorems 2.3.1 and 2.3.2. The RHS of (2.4.7) can be presented as a sum of three summands. The first summand is

$$\begin{aligned}
&\beta\left(\frac{z}{q_1qw}\right)\beta\left(\frac{w}{q_1qz}\right)(qq_1zw)^{\frac12}\\
&\times\left(w^{-\frac12}\alpha_\psi(z/w)\Psi^{*,(2)}(qq_1w)\Psi^{*,(1)}(qq_1z)\right)R(z/w)\varepsilon_z\otimes\varepsilon_wR^{-1}(z/w)\left(w^{-\frac12}\alpha_\phi(z/w)\Phi^{(2)}(w)\Phi^{(1)}(z)\right)\\
&=\frac{1}{1-z/w}\left(\frac{qq_1z}{w}\right)^{\frac12}\beta\left(\frac{z}{q_1qw}\right)\beta\left(\frac{w}{q_1qz}\right)\left(\Psi^{*,(2)}(qq_1w)\Psi^{*,(1)}(qq_1z)\right)\varepsilon_z\otimes\varepsilon_w\left(\Phi^{(2)}(w)\Phi^{(1)}(z)\right)\\
&=\frac{1}{1-z/w}\beta\left(\frac{z}{q_1qw}\right)\beta\left(\frac{q_1qz}{w}\right)\left(\Psi^*(qq_1w)\varepsilon_w\Phi(w)\right)\left(\Psi^*(qq_1z)\varepsilon_z\Phi(z)\right)\\
&=f(z/w)\left(\Psi^*(qq_1w)\varepsilon_w\Phi(w)\right)\left(\Psi^*(qq_1z)\varepsilon_z\Phi(z)\right).
\end{aligned}\tag{2.4.8}$$

Here we have used Proposition 2.3.9 and an important property

$$R(z/w)\varepsilon_z\otimes\varepsilon_wR^{-1}(z/w)=\varepsilon_z\otimes\varepsilon_w. \tag{2.4.9}$$

The second summand without factor $\beta\left(\frac{z}{q_1qw}\right)\beta\left(\frac{w}{q_1qz}\right)$ is

$$\begin{aligned}
&(qq_1zw)^{\frac12}\left(z^{-\frac12}\alpha_\psi\left(\frac{w}{z}\right)\Psi^{*,(1)}(qq_1z)\Psi^{*,(2)}(qq_1w)\right)\varepsilon_z\otimes\varepsilon_w\left((-q)^{\frac12}(q^{-1}v_-\otimes v_+-q^{-2}v_+\otimes v_-)\delta(z,q^2w)\right)(-1)^\partial\\
&=(qq_1)^{\frac12}zw\left(z^{-\frac12}\alpha_\psi(w/z)\Psi^{*,(1)}(qq_1z)\Psi^{*,(2)}(qq_1w)\right)\left((-q)^{\frac12}(q^{-1}v_-\otimes v_+-q^{-2}v_+\otimes v_-)\delta(z,q^2w)\right)(-1)^\partial\\
&=q^{-1}zw(-q_1/z)^{\frac12}\alpha_\psi(w/z)\left(q\Psi^*_-(qq_1z)\Psi^*_+(qq_1w)-\Psi^*_+(qq_1z)\Psi^*_-(qq_1w)\right)(-1)^\partial\delta(z,q^2w)\\
&=q^{-1}zw\frac{w^{-1}}{1-q^2}\delta(z,q^2w)=\frac{1}{q(1-q^2)}\delta(z/q^2w).
\end{aligned}$$

Here we used Corollary 2.3.8.

The third summand without factor $\beta\left(\frac{z}{q_1qw}\right)\beta\left(\frac{w}{q_1qz}\right)$ is

$$\begin{aligned}
&qq_1(zw)^{\frac12}\left((-q)^{\frac12}(qv_-^*\otimes v_+^*-v_+^*\otimes v_-^*)\delta(q^3q_1z,qq_1w)\right)\varepsilon_z\otimes\varepsilon_w\left(z^{-\frac12}\alpha_\phi(w/z)\Phi^{(1)}(z)\Phi^{(2)}(w)\right)(-1)^\partial\\
&=qq_1zw\left((-q)^{\frac12}(qv_-^*\otimes v_+^*-v_+^*\otimes v_-^*)\delta(q^3q_1z,qq_1w)\right)\left(z^{-\frac12}\alpha_\phi(w/z)\Phi^{(1)}(z)\Phi^{(2)}(w)\right)(-1)^\partial\\
&=-q^2q_1zw\ (-qz)^{-\frac12}\alpha_\phi(w/z)\left(q\Phi_-(z)\Phi_+(w)-\Phi_+(z)\Phi_-(w)\right)(-1)^\partial\delta(q^3q_1z,qq_1w)\\
&=-q^2q_1zw\frac{1}{zq^2(1-q^2)}\delta(q^3q_1z,qq_1w)=-\frac{1}{q(1-q^2)}\delta(q^2z/w).
\end{aligned}$$

Here we used Proposition 2.3.6

When we calculated the second and the third summands, we have omitted the multiple

$$\beta\left(\frac{q^2}{q_1 q}\right)\beta\left(\frac{q^{-2}}{q_1 q}\right)=\frac{1-q^{-2}q_1^{-1}}{1-q_1^{-1}}\left(\beta(q/q_1)\right)^2=\frac{1-q_2}{1-q_1^{-1}}\left(\beta(q/q_1)\right)^2. \tag{2.4.10}$$

So the delta-function coefficient is

$$\frac{1-q_2}{1-q_1^{-1}}\left(\beta(q/q_1)\right)^2\times\frac{1}{q(1-q^2)}=\frac{(1-q_1)(1-q_2)}{(1-q_3^{-1})}\left(\frac{\beta(q/q_1)}{q^{3/2}(q_1^{1/2}-q_1^{-1/2})}\right)^2. \tag{2.4.11}$$

To sum up, we have proven

$$\begin{aligned}&f(w/z)\left(\Psi^*(qq_1z)\varepsilon_z\Phi(z)\right)\left(\Psi^*(qq_1w)\varepsilon_w\Phi(w)\right)-f(z/w)\left(\Psi^*(qq_1w)\varepsilon_w\Phi(w)\right)\left(\Psi^*(qq_1z)\varepsilon_z\Phi(z)\right)\\&\qquad=\frac{(1-q_1)(1-q_2)}{(1-q_3^{-1})}\left(\frac{\beta(q/q_1)}{q^{3/2}(q_1^{1/2}-q_1^{-1/2})}\right)^2\left(\delta\left(\frac{z}{q^2w}\right)-\delta\left(\frac{q^2z}{w}\right)\right).\end{aligned} \tag{2.4.12}$$

Evidently, this is equivalent to the theorem. □

Connection with Verma module. *Highest weight vector* $|\lambda\rangle$ for Vir_{q_1,q_2} with *highest weight* $\lambda\in\mathbb{C}$ in a Vir_{q_1,q_2}-module is defined by the following properties

$$T_0|\lambda\rangle=\lambda|\lambda\rangle,\qquad T_n|\lambda\rangle=0\quad\text{for } n>0. \tag{2.4.13}$$

Denote by $|j\rangle=1\otimes\mathbb{C}e^{\Lambda_i+\lfloor\frac{j}{2}\rfloor\alpha}\in V_j$ the highest weight vector with respect to Heisenberg algebra.

Proposition 2.4.1. *Vector* $|j\rangle\in\mathcal{F}_u^{[j]}$ *is a highest weight vector for* Vir_{q_1,q_2} *with the highest weight*

$$\lambda_{u,j}=(-q)^{1/2}(qq_1)^{j/2}u+\left((-q)^{1/2}(qq_1)^{j/2}u\right)^{-1}. \tag{2.4.14}$$

Proof. Using (2.2.17), we obtain

$$\begin{aligned}\frac{q^{3/2}(q_1^{1/2}-q_1^{-1/2})}{\beta(q/q_1)}z^{1/2}\Psi_-^*(qq_1z)\Phi_-(z)&=\frac{q^{3/2}(q_1^{1/2}-q_1^{-1/2})}{\beta(q/q_1)}\frac{(-q^3\times qq_1z)^{-1/2}}{\beta(1/qq_1)}z^{1/2}:\Psi_-^*(qq_1z)\Phi_-(z):\\&=\frac{q^{3/2}(q_1^{1/2}-q_1^{-1/2})(-q^3\times qq_1)^{-1/2}}{(1-1/q_1)}:\Psi_-^*(qq_1z)\Phi_-(z):=(-q)^{-1/2}:\Psi_-^*(qq_1z)\Phi_-(z):.\end{aligned} \tag{2.4.15}$$

Using (2.4.15) and the formulas for explicit bosonization (2.2.8) and (2.2.12), we obtain

$$z^{1/2}\frac{q^{3/2}(q_1^{1/2}-q_1^{-1/2})}{\beta(q/q_1)}\Psi_-^*(qq_1z)\Phi_-(z)|j\rangle=(-q)^{-1/2}(qq_1)^{-j/2}|j\rangle+O(z). \tag{2.4.16}$$

here $O(z)$ is a formal power series, which vanishes at $z=0$, i.e. $\sum_{n>0}\alpha_n z^n$.

Lemma 2.4.2. *The vector* $\tilde{\pi}|j\rangle$ *coincides up to a scale with the vector* $|1-j\rangle$.

Sketch of a proof. Let us consider two grading on $V(\Lambda_0)\oplus V(\Lambda_1)$

$$\deg_{\mathrm{pr}}|j\rangle=j(j-1)/4\qquad\qquad \deg_K v=j\quad\text{iff } Kv=q^jv \tag{2.4.17}$$

$$\deg_{\mathrm{pr}}a_{-k}=k \tag{2.4.18}$$

One can check that

$$\deg_K(\pi(v))=1-\deg_K v\qquad\qquad \deg_{\mathrm{pr}}(\pi(v))=\deg_{\mathrm{pr}}v \tag{2.4.19}$$

Up to a scale, vector $|j\rangle$ is the only vector with $\deg_K=j$ and $\deg_{\mathrm{pr}}=j(j-1)/4$. □

Let us apply π-involution to (2.4.16); Proposition 2.2.2 and Lemma 2.4.2 imply

$$z^{1/2}\frac{q^{3/2}(q_1^{1/2}-q_1^{-1/2})}{\beta(q/q_1)}(-q)^{3(\frac{1}{2}-i)}\left(q_1q^3z\right)^{-\frac{1}{2}}\Psi_+^{(i,1-i),*}(qq_1z)\,(-q^3)^{i-\frac{1}{2}}z^{\frac{1}{2}}\Phi_+^{(1-i,i)}(z)|1-j\rangle$$
$$= (-q)^{-1/2}(qq_1)^{-j/2}|1-j\rangle + O(z). \quad (2.4.20)$$

Replacing of $j\mapsto 1-j$, we obtain

$$z^{1/2}\frac{q^{3/2}(q_1^{1/2}-q_1^{-1/2})}{\beta(q/q_1)}\Psi_+^*(qq_1z)\Phi_+(z)|j\rangle = (-q)^{1/2}(qq_1)^{j/2}|j\rangle + O(z). \quad (2.4.21)$$

Comparison of (2.4.16) and (2.4.21) with (2.4.5) finishes the proof. □

Verma module $M(\lambda)$ is a module with a cyclic highest weight vector $|\lambda\rangle$ and without any other relations apart from (2.4.13). Verma module $M(\lambda)$ enjoys a universal property: it maps to any module with a highest weight vector of weight λ. According to Proposition 2.4.1, there is a natural map $\phi_{u,j}\colon M(\lambda_{u,j})\to\mathcal{F}_u^{[j]}$. We will say that λ is generic if $\lambda\neq\pm(q_1^{r/2}q_2^{s/2}+q_1^{-r/2}q_2^{-s/2})$ for $r,s\in\mathbb{Z}_{\geq 1}$.

Proposition 2.4.3. *For a generic λ the Verma module $M(\lambda)$ is irreducible. The dimension of the nth graded component is $p(n)$, i.e. the number of partitions of n elements.*

This proposition follows from the fact that the determinant of the Shapovalov form for such λ is nonzero, this fact was proven in [BP98, Th. 3.3], using [SKAO96]. One can also deduce this from the irreducibility of the corresponding tensor product of Fock modules of toroidal algebra $U_{q_1,q_2,q_3}(\ddot{\mathfrak{gl}}_1)$ [FFJ+11b, Lem 3.1] and its relations to W-algebras [Neg18] [FHS+10].

We will say that a pair u,j is generic if the corresponding highest weight $\lambda_{u,j}$ is generic.

Corollary 2.4.4. *For generic values of u,j the module $\mathcal{F}_u^{[j]}$ is irreducible. The natural map $\phi_{u,j}\colon M(\lambda_{u,j})\to\mathcal{F}_u^{[j]}$ is an isomorphism.*

Proof. Note that the dimensions of the graded components of both $M(\lambda)$ and $\mathcal{F}_u^{[j]}$ equals to $p(n)$, in particular they coincide. If $M(\lambda_{u,j})$ is irreducible, then the map $\phi_{u,j}\colon M(\lambda_{u,j})\to\mathcal{F}_u^{[j]}$ is an isomorphism. □

Remark 2.4.2. As it was mentioned in Remark 2.4.1, another bosonization of Vir_{q_1,q_2} was constructed in [SKAO96]. Moreover, their formula for the highest weight essentially coincides with our formula (2.4.14). Namely, in the notation of [FF96, Sec. 3], the highest weight of the representation π_μ equals to $\lambda_{u,j}$ if q^μ in notation of *loc. cit.* equals to $(-q^3)^{\frac{1}{2}}(qq_1)^ju$ in notation of this chapter (note that parameters q,p in *loc. cit.* correspond to q_1,q_3^{-1} in this chapter). For generic u,j these modules are isomorphic since they both are isomorphic to irreducible Verma module.

The structure of the modules $\mathcal{F}_u^{[j]}$ for *non generic* values of u,j looks to be an interesting open question. Standard tool to study bosonized modules is screening operator, its construction for modules $\mathcal{F}_u^{[j]}$ is another interesting open question.

2.4.2 Twisted Deformed Virasoro algebra

Definition 2.4.2. *Twisted deformed Virasoro algebra $\mathsf{Vir}_{q_1,q_2}^{tw}$ is generated by T_r for $r\in 1/2+\mathbb{Z}$. The defining relation is*

$$\sum_{l=0}^{\infty}f_lT_{r-l}T_{s+l}-\sum_{l=0}^{\infty}f_lT_{s-l}T_{r+l} = -\frac{(1-q_1)(1-q_2)}{1-q_3^{-1}}\left(q_3^{-r}-q_3^{r}\right)\delta_{r+s,0}. \quad (2.4.22)$$

Denote $T(z)=\sum_{r\in 1/2+\mathbb{Z}} T_r z^{-r}$, $\delta_{\text{odd}}(x)=\sum_{r\in 1/2+\mathbb{Z}} x^r$ Relation (2.4.22) is equivalent to

$$f(w/z)T(z)T(w)-f(z/w)T(w)T(z)=-\frac{(1-q_1)(1-q_2)}{1-q_3^{-1}}\left(\delta_{\text{odd}}\left(\frac{w}{q_3 z}\right)-\delta_{\text{odd}}\left(\frac{q_3 w}{z}\right)\right). \quad (2.4.23)$$

Theorem 2.4.2. *The formulas below determines an action of* $\mathsf{Vir}^{tw}_{q_1,q_2}$ *on* $V(\Lambda_i)$ *for* $i=0,1$

$$T(z)=(-1)^{1/2}\frac{q^{3/2}(q_1^{1/2}-q_1^{-1/2})}{\beta(q/q_1)}\left(z\Psi^*_-(qq_1z)\Phi_+(z)+\Psi^*_+(qq_1z)\Phi_-(z)\right). \quad (2.4.24)$$

Denote the obtained representation by $\mathcal{F}^{[i]}$.

Remark 2.4.3. Note, that there is *no* twisted Virasoro algebra in a conformal field theory. This does not contradict the fact that it is possible to consider the $q,q_1\to 1$ limit of the formula 2.4.24. Let $q=e^{\hbar}, q_1=e^{\kappa\hbar}$ and assume that $T_r=\hbar\sqrt{-1/2\kappa(\kappa+2)}b_r+o(\hbar)$, then b_r satisfy twisted Heisenberg algebra $[b_r,b_s]=r\delta_{r+s,0}$. In this limit the formula (2.4.24) is standard bosoniztion of twisted Heisenberg algebra in terms of standard one (limit of a_n). In terms of representation theory of $\widehat{\mathfrak{sl}}_2$ this can be viewed as a relation between Lepowsky-Wilson and Frenkel-Kac constructions.

Proof. Let us rewrite (2.4.24) in the matrix form

$$T(z)=(-1)^{1/2}\frac{q^{3/2}(q_1^{1/2}-q_1^{-1/2})}{\beta(q/q_1)}\Psi^*(qq_1z)\varepsilon_z\Phi(z),\ \text{for } \varepsilon_z=\begin{pmatrix}0&1\\z&0\end{pmatrix}. \quad (2.4.25)$$

The proof is very similar to the proof of Theorem 2.4.1. A crucial point is that (2.4.9) holds for the new ε_z. Hence the RHS of (2.4.7) still can be presented as a sum of three summands. The first summand is still given by (2.4.8). The second summand without the factor $\beta\left(\frac{z}{q_1qw}\right)\beta\left(\frac{w}{q_1qz}\right)$

$$\begin{aligned}
&(qq_1wz)^{\frac12}\left(z^{-\frac12}\alpha_\psi(w/z)\Psi^{*,(1)}(qq_1z)\Psi^{*,(2)}(qq_1w)\right)\varepsilon_z\otimes\varepsilon_w\left((-q)^{\frac12}(q^{-1}v_-\otimes v_+-q^{-2}v_+\otimes v_-)\delta(z,q^2w)\right)(-1)^\partial\\
&=(-q_1)^{\frac12}\left(z^{-\frac12}\alpha_\psi(w/z)\Psi^{*,(1)}(qq_1z)\Psi^{*,(2)}(qq_1w)\right)\varepsilon_z\otimes\varepsilon_w\left(q^{-1}v_-\otimes v_+-q^{-2}v_+\otimes v_-\right)\delta_{\text{odd}}(z/q^2w)(-1)^\partial\\
&=(-q_1/z)^{\frac12}\alpha_\psi(w/z)\Psi^{*,(1)}(qq_1z)\Psi^{*,(2)}(qq_1w)\left(q^{-1}wv_+\otimes v_--wv_-\otimes v_+\right)\delta_{\text{odd}}(z/q^2w)(-1)^\partial\\
&=q^{-1}w\times(-q_1/z)^{\frac12}\alpha_\psi(w/z)\left(\Psi^*_+(qq_1z)\Psi^*_-(qq_1w)-q\Psi_-(qq_1z)\Psi^*_+(qq_1w)\right)\delta_{\text{odd}}(z/q^2w)(-1)^\partial\\
&=-q^{-1}w\frac{w^{-1}}{1-q^2}\delta_{\text{odd}}(z/q^2w)=-\frac{1}{q(1-q^2)}\delta_{\text{odd}}(z/q^2w).
\end{aligned}$$

The third summand without the factor $\beta\left(\frac{z}{q_1qw}\right)\beta\left(\frac{w}{q_1qz}\right)$

$$\begin{aligned}
&qq_1(wz)^{\frac12}\left((-q)^{\frac12}(qv^*_-\otimes v^*_+-v^*_+\otimes v^*_-)\delta(q^3q_1z,qq_1w)\right)\varepsilon_z\otimes\varepsilon_w\left(z^{-\frac12}\alpha_\phi(w/z)\Phi^{(1)}(z)\Phi^{(2)}(w)\right)(-1)^\partial\\
&=-(-q)^{-\frac12}\left((qv^*_-\otimes v^*_+-v^*_+\otimes v^*_-)\delta_{\text{odd}}(q^2z/w)\right)\varepsilon_z\otimes\varepsilon_w\left(z^{-\frac12}\alpha_\phi(w/z)\Phi^{(1)}(z)\Phi^{(2)}(w)\right)(-1)^\partial\\
&=-(-q)^{-\frac12}\left(qzv^*_+\otimes v^*_--wv^*_-\otimes v^*_+\right)\left(z^{-\frac12}\alpha_\phi(w/z)\Phi^{(1)}(z)\Phi^{(2)}(w)\right)\delta_{\text{odd}}(q^2z/w)(-1)^\partial\\
&=qz(-qz)^{-\frac12}\alpha_\phi(w/z)\left(q\Phi_-(z)\Phi_+(w)-\Phi_+(z)\Phi_-(w)\right)\delta_{\text{odd}}(q^2z/w)(-1)^\partial\\
&=qz\frac{1}{zq^2(1-q^2)}\delta_{\text{odd}}(q^2z/w)=\frac{1}{q(1-q^2)}\delta_{\text{odd}}(q^2z/w).
\end{aligned}$$

The end of the proof is almost the same as in non-twisted case. □

Connection with Verma module. *Highest weight vector* $|\varnothing\rangle$ in a representation of twisted deformed Virasoro algebra is defined by the following properties

$$T_r|\varnothing\rangle = 0 \quad \text{for } r > 0. \tag{2.4.26}$$

Proposition 2.4.5. *The vectors* $|\Lambda_i\rangle \in \mathcal{F}^{[i]}$ *are highest weight vectors.*

Proof. Recall the grading $\deg_{\mathrm{pr}}$ on $V(\Lambda_i)$ defined by (2.4.17) and (2.4.18). One can check that $\mathcal{F}^{[i]}$ is a graded $\mathsf{Vir}^{\mathrm{tw}}_{q_1,q_2}$-module with respect to a grading $\deg_{\mathrm{pr}} T_{-r} = r$. To finish the proof one has to note that $\deg_{\mathrm{pr}} |0\rangle = \deg_{\mathrm{pr}} |1\rangle = 0$ and $\deg_{\mathrm{pr}} |j\rangle > 0$ for $j \neq 0, 1$. $\square$

Verma module M^{tw} of twisted Virasoro algebra is a cyclic module with a cyclic vector $|\varnothing\rangle$ and without any other relations apart from (2.4.26). Verma module enjoys a universal property: it maps to any module with a highest weight vector. Hence there exist a natural map $\phi_i\colon M^{\mathrm{tw}} \to \mathcal{F}^{[i]}$ such that $|\varnothing\rangle \mapsto |\Lambda_i\rangle$.

Lemma 2.4.6. *Verma module* M^{tw} *is spanned by*

$$T_{-r_m} \dots T_{-r_1}|\varnothing\rangle \quad \textit{for } 0 < r_1 \leq r_2 \leq \dots \leq r_m. \tag{2.4.27}$$

Sketch of a proof. One can prove that any element $T_{s_1} \dots T_{s_k}|\varnothing\rangle$ can be presented as a linear combination of vectors (2.4.27) using (2.4.22) by an induction. $\square$

Proposition 2.4.7. *For generic* q_3 *the Verma module* M^{tw} *is irreducible. Natural maps* $\phi_i\colon M^{tw} \to \mathcal{F}^{[i]}$ *are isomorphisms.*

Proof. The representation $\mathcal{F}^{[i]}$ for $q_3 = 1$ was considered in Example 1.7.1; it follows from Section 1.7.3 that the representations are irreducible. Hence $\mathcal{F}^{[i]}$ is irreducible for generic q_3. Then the maps $\phi_i\colon M^{\mathrm{tw}} \to \mathcal{F}^{[i]}$ are surjective. Now recall Gauss identity

$$\prod_{r=\frac{1}{2}+\mathbb{Z}_{\geq 0}} \frac{1}{1-\mathfrak{q}^r} = \sum_{j \in i+2\mathbb{Z}} \frac{\mathfrak{q}^{\frac{j(j-1)}{4}}}{\prod_{n=1}^{\infty}(1-\mathfrak{q}^n)} \quad \text{for } i = 0, 1. \tag{2.4.28}$$

According to Lemma 2.4.6, the dimensions of the graded components of M^{tw} do not exceed the corresponding coefficient of the LHS of (2.4.28). On the other hand, the coefficients of the RHS of (2.4.28) are equal to the dimensions of the graded components of $\mathcal{F}^{[i]}$. Hence, it follows from surjectivity of ϕ_i that ϕ_i is an isomorphism. $\square$

Corollary 2.4.8. *For generic* q_3*, the vectors* (2.4.27) *form a basis of* M^{tw}*.*

Chapter 3

Semi-infinite construction

3.1 Double Affine Hecke Algebra

In this section we recall the definition and basic properties of double affine Hecke algebra (DAHA) [Che92, Kir97, Che05]. This section consists no new results.

Definition 3.1.1. The DAHA for $\mathfrak{gl}_N$ is an algebra $\mathcal{H}_N$ with generators $T_1, \ldots, T_{N-1}, \pi^{\pm 1}, Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}$ and relations

$$(T_i - v)(T_i + v^{-1}) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \tag{3.1.1}$$
$$T_i Y_i T_i = Y_{i+1}, \quad T_i Y_j = Y_j T_i, j \neq i, i+1 \tag{3.1.2}$$
$$\pi Y_i \pi^{-1} = q^{\delta_{i,N}} Y_{i+1}, \quad Y_i Y_j = Y_j Y_i \tag{3.1.3}$$
$$\pi T_i \pi^{-1} = T_{i+1}, \quad \pi^N T_i = T_i \pi^N \tag{3.1.4}$$

Here and below we use the convention $Y_1 = Y_{N+1}$.

The operators $T_1, \ldots, T_{N-1}$ generate finite Hecke algebra H. The operators $T_1, \ldots, T_{N-1}, Y_1, \ldots, Y_N$ generate affine Hecke algebra H^Y. The operators $T_1, \ldots, T_{N-1}, \pi$ generate affine Hecke algebra H^X, here one can define

$$X_i = T_i \ldots T_{N-1} \pi^{-1} T_1^{-1} \ldots T_{i-1}^{-1}. \tag{3.1.5}$$

The relations on X_i are

$$T_i^{-1} X_i T_i^{-1} = X_{i+1}, \quad X_2^{-1} Y_1 X_2 Y_1^{-1} = T_1^2, \quad X_1 Y_2^{-1} X_1^{-1} Y_2 = T_1^2 \tag{3.1.6}$$

Let $\widetilde{SL}(2,\mathbb{Z})$ be the braid group on 3 stands. More precisely, $\widetilde{SL}(2,\mathbb{Z})$ is generatred by τ_+ and τ_- with the relation $\tau_+ \tau_-^{-1} \tau_+ = \tau_-^{-1} \tau_+ \tau_-^{-1}$. The reason for our notation is that $\widetilde{SL}(2,\mathbb{Z})$ is an extensions of $SL(2,\mathbb{Z})$ by $\mathbb{Z}$, the projection is given by

$$\tau_+ \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \tau_- \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{3.1.7}$$

The kernal is generated by $(\tau_+ \tau_-^{-1} \tau_+)^4$.

Proposition 3.1.1. *There is an action of $\widetilde{SL}(2,\mathbb{Z})$ on $\mathcal{H}_N$ determined by the following formulas*

$$\tau_+ : \quad T_i \to T_i, \;\; X_i \to X_i, \;\; Y_i \to Y_i X_i T_{i-1} \ldots T_1 T_1 \ldots T_{i-1} \tag{3.1.8}$$
$$\tau_- : \quad T_i \to T_i, \;\; X_i \to X_i Y_i T_{i-1}^{-1} \ldots T_1^{-1} T_1^{-1} \ldots T_{i-1}^{-1}, \;\; Y_i \to Y_i \tag{3.1.9}$$

The algebra $\mathcal{H}_N$ is bigraded with obvious gradings $\deg_X$ and $\deg_Y$ defined by

$$\deg_X \pi = -1 \qquad \deg_X Y_i = 0 \qquad \deg_X T_i = 0 \tag{3.1.10}$$
$$\deg_Y \pi = 0 \qquad \deg_Y Y_i = 1 \qquad \deg_Y T_i = 0. \tag{3.1.11}$$

Lemma 3.1.1. *Let $n, n' \in \mathbb{Z}_{\geq 0}$ be coprime integers. Let $m, m' \in \mathbb{Z}_{\geq 0}$ be unique pair of integers such that $m < n$, $m' < n'$ and $nm' - n'm = 1$. Then, there is an $\widetilde{SL}(2, \mathbb{Z})$ transformation such that X_i^{-1} is mapped to $\mathbf{B}_i$ and Y_i is mapped to $\mathbf{A}_i$ such that*

$$\deg_X(\mathbf{B}_i) = -n,\ \deg_Y(\mathbf{B}_i) = n' \quad \deg_X(\mathbf{A}_i) = -m,\ \deg_Y(\mathbf{A}_i) = m' \tag{3.1.12}$$

Moreover $\mathbf{B}_{i+1} = T_i \dots T_1 \mathbf{B}_1 T_1 \dots T_i$, $\mathbf{A}_{i+1} = T_i \dots T_1 \mathbf{A}_1 T_1 \dots T_i$, *and*

$$\mathbf{B}_1 = Z_1 \cdot \dots Z_{n+n'}, \quad \mathbf{A}_1 = W_1 \cdot \dots W_{m+m'}, \tag{3.1.13}$$

where

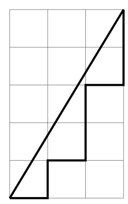
$$Z_j = Y_1 \ \ \text{if} \ \left\lfloor \frac{jn}{n+n'} \right\rfloor = \left\lfloor \frac{(j-1)n}{n+n'} \right\rfloor, \qquad Z_j = X_1^{-1} \ \ \text{if} \ \left\lfloor \frac{jn}{n+n'} \right\rfloor = \left\lfloor \frac{(j-1)n}{n+n'} \right\rfloor + 1 \tag{3.1.14}$$

$$W_j = Y_1 \ \ \text{if} \ \left\lfloor \frac{jm}{m+m'} \right\rfloor = \left\lfloor \frac{(j-1)m}{m+m'} \right\rfloor, \qquad W_j = X_1^{-1} \ \ \text{if} \ \left\lfloor \frac{jm}{m+m'} \right\rfloor = \left\lfloor \frac{(j-1)m}{m+m'} \right\rfloor + 1. \tag{3.1.15}$$

Proof. The formulas for $\mathbf{B}_i$, $\mathbf{A}_i$, in terms of $\mathbf{B}_1$, $\mathbf{A}_1$ follows from the relations $X_{j+1}^{-1} = T_j X_j^{-1} T_j$, $Y_{j+1} = T_j Y_j T_j$. The formulas for $\mathbf{B}_1$,$\mathbf{A}_1$, can be proven using Euclidean algorithm. For the step one uses the formulas $\tau_+^{-1}(X_1^{-1}) = Y_1 X_1^{-1} = \tau_-^{-1}(Y_1)$. □

Euclidean algorithm used in the proof can be viewed as decomposition of the matrix $\begin{pmatrix} n & n' \\ m & m' \end{pmatrix}$ into product of the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ which correspond to τ_+^{-1}, τ_-^{-1} in the basis $(-1, 0)$, $(0, 1)$.

The formula for $\mathbf{B}_1$ has also following geometric interpretation. Draw the segment from $(0, 0)$ to (n', n) and draw closest line consisting of horizontal and vertical lines below. Then, for any horizontal segment we write Y_1 and for any vertical segment we write X_1^{-1}.



Example 3.1.1. Let us take $n = 5$ and $n' = 3$. Then

$$\begin{aligned} \tau_-^1 \tau_+^{-1} \tau_-^{-1} \tau_+^{-1}(X_1^{-1}) &= Y_1 X_1^{-1} Y_1 X_1^{-2} Y_1 X_1^{-2}, \\ \tau_-^1 \tau_+^{-1} \tau_-^{-1} \tau_+^{-1}(Y_1) &= Y_1 X_1^{-1} Y_1 X_1^{-2} \end{aligned}$$

The formulas of $\mathbf{B}_1$ agrees with the form of the sequence $\frac{jn}{n+n'}$ namely $0, \frac{5}{8}, \frac{10}{8}, \frac{15}{8}, \frac{20}{8}, \frac{25}{8}, \frac{30}{8}, \frac{35}{8}, \frac{40}{8}$ as well as geometric description.

3.1.1 Cherednik representation

The algebra H^X has *trivial* 1-dimensional representation

$$T_i \mapsto v, \quad \pi \mapsto u.$$

Cherednik representation $\mathbf{C}_u$ is a representation of $\mathcal{H}_N$ which is induced from the trivial representation of H^X. It can be identified with the space of Laurent polynomials $\mathbb{C}[Y_1^{\pm 1}, \dots, Y_N^{\pm 1}]$. The action of the generators T_i and π can be written as

$$T_i = s_i^Y + (v - v^{-1}) \frac{s_i^Y - 1}{Y_i / Y_{i+1} - 1}, \qquad \pi(Y_1^{\lambda_1} Y_2^{\lambda_2} \dots Y_N^{\lambda_N}) = u q^{\lambda_N} Y_1^{\lambda_N} Y_2^{\lambda_1} Y_3^{\lambda_2} \dots Y_N^{\lambda_{N-1}},$$

here s_i^Y is permutation if Y_i and Y_{i+1}.

For any composition $\lambda = (\lambda_1, \cdots \lambda_N) \in \mathbb{Z}^N$ we denote $Y^\lambda = Y_1^{\lambda_1} \cdot \dots Y_N^{\lambda_N}$, such vectors form the standard monomial basis in Cherednik representation. We will say $\lambda \leq \mu$ if $\mu - \lambda \in \oplus \mathbb{Z}_{\geq 0} \alpha_i$, where α_i are positive simple roots of $\mathfrak{sl}_N$.

Definition 3.1.2. Let λ, $\mu \in \mathbb{Z}^N$. We write $\lambda \prec \mu$ if

1. $\lambda^+ < \mu^+$ where λ^+ is the dominant coweight lying in the orbit of λ, and similarly for μ^+
2. $\lambda^+ = \mu^+$ and $\lambda < \mu$

For example, for $N = 2$

$$(1,1) \prec (0,2) \prec (2,0)$$

The action of the operators $X_1, \dots, X_N$ in the basis Y^λ is triangular with respect to the order $\prec$. More explicitly

$$X_i Y^\lambda = u^{-1} q^{-\lambda_i} v^{-\mathbb{1}_i^{(\lambda)}} Y^\lambda + \sum_{\mu \prec \lambda} x_{\lambda,\mu} Y^\mu, \tag{3.1.16}$$

here

$$\mathbb{1}_i^{(\lambda)} = \big|\{j|\lambda_i < \lambda_j\}\big| + \big|\{j|i > j, \lambda_i = \lambda_j\}\big| - \big|\{j|\lambda_i > \lambda_j\}\big| - \big|\{j|i < j, \lambda_i = \lambda_j\}\big| \tag{3.1.17}$$

The non-symmetric Macdonald polynomials E_λ are defined as eigenvectors of $X_1, \dots, X_N$ with leading term Y^λ. Note that (3.1.16) imply that

$$X_i E_\lambda = u^{-1} q^{-\lambda_i} v^{-\mathbb{1}_i^{(\lambda)}} E_\lambda. \tag{3.1.18}$$

3.2 Representation

In this section we introduce representation $\mathbf{C}^{(n,n_{tw})}_{u_0,\dots,u_{n-1}}$, generalize (3.1.16) and interpret the obtained representation as twisted Cherednik representation.

3.2.1 Explicit construction

Action of affine Hecke algebra Fix n and let $\mathbb{C}^n$ be a vector space with the basis $e_0, \dots, e_{n-1}$. Define an R matrix acting on $\mathbb{C}^n \otimes \mathbb{C}^n$

$$R = \sum_a v E_{aa} \otimes E_{a,a} + \sum_{a<b} \Big(E_{ab} \otimes E_{ba} + E_{ba} \otimes E_{ab} + (v - v^{-1}) E_{aa} \otimes E_{bb} \Big). \tag{3.2.1}$$

Define an action of H on $(\mathbb{C}^n)^{\otimes N}$ by the formula $T_i \mapsto R_{i,i+1}$, here indices encodes factors on which R-matrix acts. One can induce an action of H^Y on $(\mathbb{C}^n)^{\otimes N}[Y_1^{\pm 1}, \dots Y_N^{\pm 1}]$.

One can write the action of T_i explicitly. Let s_i^Y be an operator acting on $(\mathbb{C}^n)^{\otimes N}[Y_1^{\pm 1}, \dots, Y_N^{\pm 1}]$ which swaps Y_i and Y_{i+1}. Let s_i^e be an operator acting on $(\mathbb{C}^n)^{\otimes N}[Y_1^{\pm 1}, \dots, Y_N^{\pm 1}]$ which swaps tensor factors number i and $i+1$ (and commutes with all Y_i). Finally, let $s_i = s_i^Y s_i^e$. Then the action of T_i is given by the following formula

$$T_i = s_i^Y R_{i,i+1} + (v - v^{-1}) \frac{s_i^Y - 1}{Y_i / Y_{i+1} - 1} \tag{3.2.2}$$

The obtained representation of affine Hecke algebra is well-know [GRV94], [CP94]. It appears in the context of quantum affine Shcur-Weyl duality.

Action of DAHA Below we will use identification

$$\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \to (\mathbb{C}^n)^{\otimes N}[Y_1^{\pm 1}, \dots, Y_N^{\pm 1}] \tag{3.2.3}$$

$$(Y^{j_1}e_{i_1}) \otimes \dots \otimes (Y^{j_N}e_{i_N}) \mapsto Y_1^{j_1} \dots Y_n^{j_n} e_{i_1} \otimes \dots \otimes e_n \tag{3.2.4}$$

Let us introduce $e_i \in \mathbb{C}^n[Y^{\pm 1}]$ for $i \in \mathbb{Z}$ by setting

$$e_i = Y^{-1}e_{i+n}. \tag{3.2.5}$$

Introduce operators κ and D acting on $\mathbb{C}^n[Y^{\pm 1}]$ by $\kappa e_i = e_{i-1}$, $D(Y^j e_a) = u_a q^j Y^j e_a$ for $a = 0, \dots, n-1$. By κ_i and D_i we denote the corresponding operators acting by κ and D on i-th tensor factor.

Theorem 3.2.1. *For any $n_{tw} \in \mathbb{Z}$, there is an action of algebra $\mathcal{H}_N$ on $(\mathbb{C}^n)^{\otimes N}[Y_1^{\pm 1}, \dots, Y_N^{\pm 1}]$ determined by the following conditions*

- *subalgebra H^Y acts as discribed above*
- $\pi = \kappa_1^{n_{tw}} D_1 s_1 \dots s_{N-1}$

Denote the obtained representation by $\mathbf{C}^{(n,n_{tw})}_{u_0,\dots,u_{n-1}}$.

Proof. It is enough to check the relations which involve π. The relations $\pi Y_i \pi^{-1} = Y_{i+1}$ and $\pi T_i \pi^{-1} = T_{i+1}$ are easy to see. Let us check that π^N commutes with T_i. Since

$$\pi^N(w_1 \otimes \dots \otimes w_N) = \kappa^{n_{tw}} D w_1 \otimes \kappa^{n_{tw}} D w_2 \otimes \dots \otimes \kappa^{n_{tw}} D w_N$$

it is sufficient to consider only the $N = 2$ case. In this case we denote $T = T_1$ for brevity. Let $l = m + nk + s$ for $k \geq 0$ and $s = 0, \dots n-1$. For $s = 0$

$$T(e_m \otimes e_l) = v e_l \otimes e_m + (v - v^{-1}) \sum_{j=1}^{k} e_{l-nj} \otimes e_{m+nj} \qquad \text{for } k \geq 0 \tag{3.2.6}$$

$$T(e_l \otimes e_m) = v^{-1} e_m \otimes e_l - (v - v^{-1}) \sum_{j=1}^{k-1} e_{l-nj} \otimes e_{m+nj} \qquad \text{for } k > 0 \tag{3.2.7}$$

For $s > 0$

$$T(e_m \otimes e_l) = e_l \otimes e_m + (v - v^{-1}) \sum_{j=0}^{k} e_{m+nj} \otimes e_{l-nj}, \tag{3.2.8}$$

$$T(e_l \otimes e_m) = e_m \otimes e_l - (v - v^{-1}) \sum_{j=1}^{k} e_{l-nj} \otimes e_{m+nj}. \tag{3.2.9}$$

Since the formulas (3.2.6)–(3.2.9) are invariant under the shift $l \mapsto l - n_{tw}$, $m \mapsto m - n_{tw}$, we see that $T\pi^2 = \pi^2 T$ for $N = 2$. □

3.2.2 Triangularity of Macdonald operators

Introduce a grading on $(\mathbb{C}^n[Y^{\pm 1}])^{\otimes N}$ as follows

$$\deg e_{a_1} \otimes \dots \otimes e_{a_N} = \sum a_i, \qquad \deg Y_i = n. \tag{3.2.10}$$

Then the operators T_i preserve the grading and $\deg \pi = -n_{tw}$. Hence generators $\deg X_i = n_{tw}$.

To simplify our notation, let us assume $n, n_{tw} > 0$. Let $d = \gcd(n, n_{tw})$, we use notations $n'' = n/d$, $n' = n_{tw}/d$. Let m, m' such numbers that $nm' - n_{tw}m = d$, $0 \le m' < n'$, $0 \le m < n''$. Hence there is $\sigma \in \widetilde{SL}(2,\mathbb{Z})$ such that

$$\sigma(X_i) = \mathbf{B}_i^{-1} \qquad \deg_X(\mathbf{B}_i) = -n'',\ \deg_Y(\mathbf{B}_i) = n', \qquad \deg \mathbf{B}_i = 0 \tag{3.2.11}$$

$$\sigma(Y_i) = \mathbf{A}_i \qquad \deg_X(\mathbf{A}_i) = -m,\ \deg_Y(\mathbf{A}_i) = m', \qquad \deg \mathbf{A}_i = d \tag{3.2.12}$$

The corresponding matrix in $SL(2,\mathbb{Z})$ is $\begin{pmatrix} n'' & -m \\ -n' & m' \end{pmatrix}$. Lemma 3.1.1 gives explicit formulas for $\mathbf{B}_i, \mathbf{A}_i$.

For any collection $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N$, we can consider a vector $e_\lambda = e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_N}$.

Consider an operator $G_i = T_i s_i$. It follows from (3.2.2) that

$$G_i = R_{i,i+1} s_i^e + (v - v^{-1}) \frac{1 - s_i^Y}{Y_i/Y_{i+1} - 1} s_i^e \tag{3.2.13}$$

Denote $G_{i,i+1} = G_i$. For any $i \neq j$ let $G_{i,j}$ be the operator given by the formula (3.2.13) with $i, i+1$ replaced by i, j correspondingly. Using this notation, we can write the following formula

$$X_1^{-1} = \pi T_{N-1}^{-1} \dots T_1^{-1} = \kappa_1^{n_{tw}} D_1 s_1 \dots s_{N-1} s_{N-1} G_{N-1,N}^{-1} \dots s_1 G_{1,2}^{-1} = \kappa_1^{n_{tw}} D_1 G_{1,N}^{-1} \dots G_{1,2}^{-1} \tag{3.2.14}$$

Note also that $Y_1 = \kappa_1^{-n}$

We are going to prove that operators $\mathbf{B}_i$ are triangular in the basis e_λ. The prove is just a computation. Let us first explain the idea on the following example (cf. Example 3.1.1).

Example 3.2.1. Let us take $N = 3$, $n = 5$, $n_{tw} = 3$. Then we have

$$\begin{aligned}\mathbf{B}_1 &= Y_1 X_1^{-1} Y_1 X_1^{-2} Y_1 X_1^{-2} = \kappa_1^{-5} (\kappa_1^3 D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^3 D_1 G_{1,3}^{-1} G_{1,2}^{-1})^2 \kappa_1^{-5} (\kappa_1^3 D_1 G_{1,3}^{-1} G_{1,2}^{-1})^2 \\ &= (\kappa_1^{-2} D_1 G_{1,3}^{-1} G_{1,2}^{-1} \kappa_1^2)(\kappa_1^{-4} D_1 G_{1,3}^{-1} G_{1,2}^{-1} \kappa_1^4)(\kappa_1^{-1} D_1 G_{1,3}^{-1} G_{1,2}^{-1} \kappa_1^1)(\kappa_1^{-3} D_1 G_{1,3}^{-1} G_{1,2}^{-1} \kappa_1^3)(D_1 G_{1,3}^{-1} G_{1,2}^{-1})\end{aligned}$$

$$\begin{aligned}\mathbf{B}_2 &= T_1 \mathbf{B}_1 T_1 = G_{1,2} s_1 \kappa_1^{-5} (\kappa_1^3 D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^3 D_1 G_{1,3}^{-1} G_{1,2}^{-1})^2 \kappa_1^{-5} (\kappa_1^3 D_1 G_{1,3}^{-1} G_{1,2}^{-1})^2 G_{1,2} s_1 \\ &= G_{1,2} (\kappa_2^{-2} D_2 G_{2,3}^{-1} G_{2,1}^{-1} \kappa_2^2)(\kappa_2^{-4} D_2 G_{2,3}^{-1} G_{2,1}^{-1} \kappa_2^4)(\kappa_2^{-1} D_2 G_{2,3}^{-1} G_{2,1}^{-1} \kappa_2^1)(\kappa_2^{-3} D_2 G_{2,3}^{-1} G_{2,1}^{-1} \kappa_2^3) D_2 G_{2,3}^{-1}\end{aligned}$$

$$\begin{aligned}\mathbf{B}_3 &= T_2 T_1 \mathbf{B}_1 T_1 T_2 \\ &= G_{2,3} s_2 G_{1,2} s_1 \kappa_1^{-5} (\kappa_1^3 D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^3 D_1 G_{1,3}^{-1} G_{1,2}^{-1})^2 \kappa_1^{-5} (\kappa_1^3 D_1 G_{1,3}^{-1} G_{1,2}^{-1})^2 G_{1,2} s_1 G_{2,3} s_2 \\ &= G_{2,3} G_{1,3} (\kappa_3^{-2} D_3 G_{3,2}^{-1} G_{3,1}^{-1} \kappa_3^2)(\kappa_3^{-4} D_3 G_{3,2}^{-1} G_{3,1}^{-1} \kappa_3^4)(\kappa_3^{-1} D_3 G_{3,2}^{-1} G_{3,1}^{-1} \kappa_3^1)(\kappa_3^{-3} D_3 G_{3,2}^{-1} G_{3,1}^{-1} \kappa_3^3) D_3\end{aligned}$$

Using Proposition 3.2.2 below, we see that all these operators are triangular.

Now we proceed to the proof.

Proposition 3.2.1. *The operator G_i is triangular in the basis e_λ with respect to order $\prec$.*

Proof. It is sufficient to consider case $N = 2$. In this case we will write simply G omitting the index. The formulas below is just a reformulation of (3.2.6)–(3.2.9). Recall that $l = m + nk + s$ for $k \ge 0$ and $s = 0, \dots n-1$. For $s = 0$

$$G(e_l \otimes e_m) = v e_l \otimes e_m + (v - v^{-1}) \sum_{j=1}^{k} e_{l-nj} \otimes e_{m+nj} \qquad \text{for } k \ge 0, \tag{3.2.15}$$

$$G(e_m \otimes e_l) = v^{-1} e_m \otimes e_l - (v - v^{-1}) \sum_{j=1}^{k-1} e_{l-nj} \otimes e_{m+nj} \qquad \text{for } k > 0. \tag{3.2.16}$$

For $s > 0$

$$G(e_l \otimes e_m) = e_l \otimes e_m + (v - v^{-1}) \sum_{j=0}^{k} e_{m+nj} \otimes e_{l-nj}, \tag{3.2.17}$$

$$G(e_m \otimes e_l) = e_m \otimes e_l - (v - v^{-1}) \sum_{j=1}^{k} e_{l-nj} \otimes e_{m+nj}. \tag{3.2.18}$$

□

Proposition 3.2.2. *For $i < j$ operators $\kappa_i^{-d} G_{i,j} \kappa_i^d$, $0 \le d < n$ and $\kappa_j^{-d} G_{j,i} \kappa_j^d$, $0 < d < n$ are triangular in the basis e_λ with respect to order $\prec$. Operators $\kappa_i^d D_i \kappa_i^d$ are diagonal for any d.*

Proof. It is sufficient to consider $N = 2$ and operators $\kappa_1^{-d} G \kappa_1^d$, $0 \le d < n$ and $s_1 \kappa_1^{-d} G \kappa_1^d s_1$, $0 < d < n$. Everything follows from (3.2.15)–(3.2.18). □

Theorem 3.2.2. *Operators $\mathbf{B}_1, \dots \mathbf{B}_N$ are triangular in the basis e_λ with respect to order $\prec$.*

Proof. Recall that $n'' = n/d$, $n' = n_{tw}/d$. Using Lemma 3.1.1 we can write

$$\mathbf{B}_i = T_{i-1} \dots T_1 Z_1 \dots Z_{n''+n'} T_1 \dots T_{i-1}. \tag{3.2.19}$$

Now we substitute formulas for $Y_1 = \kappa_1^{-n}$, $X_1^{-1} = \kappa_1^{n_{tw}} D_1 G_{1,N}^{-1} \dots G_{1,2}^{-1}$ and

$$T_{i-1} \dots T_1 = G_{i-1,i} \dots G_{1,i} s_{i-1} \dots s_1,$$
$$X_1^{-1} T_1 \dots T_{i-1} = \kappa^{n_{tw}} D_1 s_1 \dots s_{i-1} s_{N-1} G_{i,N}^{-1} \dots G_{i,i+1}^{-1}.$$

Hence we get

$$\mathbf{B}_i = G_{i-1,i} \dots G_{1,i} \Bigg(\prod_{\substack{j<n'+n'' \\ Z_j = X_1^{-1}}} \kappa_i^{-d_j} D_i G_{i,N}^{-1} \dots G_{i,1}^{-1} \kappa_i^{d_j} \Bigg) D_i G_{i,N}^{-1} \dots G_{i,i+1}^{-1}. \tag{3.2.20}$$

Let $\{x\}$ denote the fractional part of $x \in \mathbb{R}$. One can observe that

$$\begin{aligned} d_j &= n \big| \{ s | s < j, Z_s = Y_1 \} \big| - n_{tw} \big| \{ s | s \le j, Z_s = X_1^{-1} \} \big| \\ &= n \left(j - \left\lfloor \frac{jn''}{n''+n'} \right\rfloor \right) - n_{tw} \left\lfloor \frac{jn''}{n''+n'} \right\rfloor = (n + n_{tw}) \left\{ \frac{jn''}{n''+n'} \right\}. \end{aligned} \tag{3.2.21}$$

Here j is such that $Z_j = X_1^{-1}$, hence $\left\{ \frac{jn''}{n''+n'} \right\} < \frac{n''}{n''+n'}$ by the condition (3.1.14). Hence $0 < d_j < n$. Therefore Proposition 3.2.2 imply, that the operator is upper-triangular. □

Corollary 3.2.3. *There are eigenvectors $\tilde{E}_\lambda = e_\lambda + \sum_{\mu \prec \lambda} \beta_{\lambda,\mu} e_\mu$ of $\mathbf{B}_1, \dots, \mathbf{B}_N$ in $\mathbf{C}^{(n,n')}_{u_0,\dots,u_{n-1}}$. The eigenvalues are given by*

$$\mathbf{B}_i \tilde{E}_\lambda = u_0 \dots u_{n-1} q^{1-n} v^{1_i^{(\lambda)}} q^{\lambda_i} \tilde{E}_\lambda \tag{3.2.22}$$

Proof. Theorem 3.2.2 is equivalent to the following formula

$$\mathbf{B}_i e_\lambda = b_{\lambda,\lambda} e_\lambda + \sum_{\mu \prec \lambda} b_{\lambda,\mu} e_\mu \tag{3.2.23}$$

Now we compute $b_{\lambda,\lambda}$ using the formula (3.2.20). It follows from the computation in the proof of Theorem that the numbers d_j are distinct and form a set $\{1, \dots, n-1\}$. Hence it remains to compute diagonal term in the action of the operators

a) $G_{1,i}$ for $j < i$; *b)* $\kappa_i^{-d} G_{i,N}^{-1} D_i \kappa_i^d$, $\kappa_i^{-d} G_{i,j}^{-1} \kappa_i^d$ for $0 < d < n, j \ne i$ *c)* $G_{i,j}^{-1}$ for $j > i$.

We get

$$b_{\lambda,\lambda} = v^{|\{j<i|\lambda_j\geq\lambda_i,\lambda_j\equiv\lambda_i\}|-|\{j<i|\lambda_j<\lambda_i,\lambda_j\equiv\lambda_i\}|}v^{|\{j<i|\lambda_j>\lambda_i,\lambda_j\not\equiv\lambda_i\}|-|\{j<i|\lambda_j\leq\lambda_i,\lambda_j\not\equiv\lambda_i\}|}$$
$$(u_0\dots u_{n-1})q^{\lambda_i+1-n}v^{|\{j>i|\lambda_j>\lambda_i\}|-|\{j>i|\lambda_j\leq\lambda_i\}|} = u_0\dots u_{n-1}q^{1-n}v^{1_i^{(\lambda)}}q^{\lambda_i}$$

where $\equiv$ stands for $\equiv \pmod n$ and we used formulas (3.2.15)-(3.2.18). □

3.2.3 Monomial basis

Proposition 3.2.4. *Operators $\mathbf{A}_i$ can be represented in the form $\mathbf{A}_i = \mathbf{A}'_i\kappa_i^{-d}\mathbf{A}''_i$, where the operators $\mathbf{A}'_i$, $\mathbf{A}''_i$ are triangular with respect to the order $\prec$. Moreover, matrix entries of $\mathbf{A}'_i$, $\mathbf{A}''_i$ on the diagonal are monomials.*

The proof is similar to the proof of Theorem 3.2.2.

Proof. Using (3.1.13) we get

$$\begin{aligned}\mathbf{A}_i &= T_{i-1}\dots T_1W_1\dots W_{m+m'}T_1\dots T_{i-1}\\ &= G_{i-1,i}\dots G_{1,i}\Bigg(\prod_{j<m+m',W_j=X_1^{-1}}\kappa_i^{-c_j}D_iG_{i,N}^{-1}\dots G_{i,1}^{-1}\kappa_i^{c_j}\Bigg)\kappa_i^{-d}D_iG_{i,N}^{-1}\dots G_{i,i+1}^{-1}.\end{aligned}\tag{3.2.24}$$

It remains to compute numbers c_j. We have

$$\begin{aligned}c_j &= n\big|\{s|s<j,W_s=Y_1\}\big| - n_{tw}\big|\{s|s\leq j,W_s=X_1^{-1}\}\big|\\ &= n\left(j-\left\lfloor\frac{jm}{m+m'}\right\rfloor\right) - n_{tw}\left\lfloor\frac{jm}{m+m'}\right\rfloor = d\left((n''+n')\left\{\frac{jm}{m+m'}\right\}+\frac{j}{m+m'}\right),\end{aligned}\tag{3.2.25}$$

Since $W_j = X_1^{-1}$ we have $0\leq\left\{\frac{jm}{m+m'}\right\}\leq\frac{m-1}{m+m'}$. Hence

$$(n''+n')\left\{\frac{jm}{m+m'}\right\}\leq(n''+n')\frac{m-1}{m+m'} = \frac{n''m+n''m'-1-n''-n'}{m+m'} < n''-1$$

Hence $0<c_j<n$. Using Proposition 3.2.2 we conclude the proof. □

Let us now assume that $d=1$, i.e. $n=n''$, $n_{tw}=n'$. In this case, operators $\mathbf{A}_i$ increase grading by 1. For any $\lambda\in\mathbb{Z}^{\mathbb{N}}$ denote

$$A_\lambda = \mathbf{A}_1^{\lambda_1}\dots\mathbf{A}_N^{\lambda_N}e_0\otimes\dots\otimes e_0 \tag{3.2.26}$$

Theorem 3.2.3. *Vectors A_λ form a basis in $(\mathbb{C}^n[Y^{\pm1}])^{\otimes N}$ and this basis is triangular with respect to e_λ basis. Moreover, we have*

$$A_\lambda = \alpha_{\lambda,\lambda}e_\lambda + \sum_{\mu\prec\lambda}\alpha_{\lambda,\mu}e_\mu \tag{3.2.27}$$

where the coefficients $\alpha_{\lambda,\mu}\in\mathbb{Z}[q^{\pm1},v^{\pm1}]$ and $\alpha_{\lambda,\lambda}$ is invertible in $\mathbb{Z}[q^{\pm1},v^{\pm1}]$.

Proof. Since $\mathbf{A}_i$ is expressed via operators $T_j^{\pm1},Y_j^{\pm1},\pi^{\pm1}$ its matrix elements in e_λ basis belong to $\mathbb{Z}[q^{\pm1},v^{\pm1}]$. Hence, the vectors A_λ expand in e_λ basis with coefficients in $\mathbb{Z}[q^{\pm1},v^{\pm1}]$.

The product $\mathbf{A}_1\dots\mathbf{A}_N$ is some combination of products $Y_1\dots Y_N=\kappa_1^{-n}\dots\kappa_N^{-n}$ and $X_1^{-1}\dots X_N^{-1} = \pi^N = \kappa_1^{n'}\dots\kappa_N^{n'}D_1\dots D_N$. Hence the product $\mathbf{A}_1\dots\mathbf{A}_N$ is diagonal in the e_λ basis. Therefore it is sufficient to prove the formula (3.2.27) for the compositions such that $\lambda_1,\dots,\lambda_N\geq0$.

Let $l = \max \lambda_j$. We proceed by induction on l. For $l = 0$ there is nothing to prove. The induction step is $l - 1 \to l$. Let $1 \le i_1 < \cdots < i_k \le N$ be a subset of indices such that $\lambda_{i_s} = l$ and $\lambda_j < l$, for $j \notin \{i_1, \dots, i_k\}$. Let $\lambda^{(s)}$, $0 \le s \le k$ be a composition such that

$$\lambda_j^{(s)} = \begin{cases} l-1, & \text{for } j = i_1, \dots i_s \\ l, & \text{for } j = i_{s+1}, \dots i_k \\ \lambda_j, & \text{for } j \notin \{i_1, \dots, i_k\} \end{cases}$$

For example $\lambda^{(0)} = \lambda$.

By the induction hypothesis we know that $A_{\lambda^{(k)}}$ is a linear combination of e_μ with $\mu \prec \lambda^{(k)}$ and coefficient $\alpha_{\lambda^{(k)},\lambda^{(k)}}$ is invertible. Now we prove by induction on s that

$$A_{\lambda^{(s)}} = \mathbf{A}_{i_{s+1}} \dots \mathbf{A}_{i_k} A_{\lambda^{(k)}}$$

satisfy condition (3.2.27) with additional constraint that for all terms in right side $\mu_j < l$, for $l < i_{s+1}$. The induction base is $s = k$, the induction step is $s \mapsto s - 1$ and follows from the formula (3.2.24). Namely, by the Proposition 3.2.2 all triangular terms in the formula (3.2.24) have invertible elements on the diagonal and cannot make $\mu_j = l$ for $j < i_s$. $\square$

Corollary 3.2.5. *The elements e_λ expand in monomial basis A_λ with coefficients in $\mathbb{Z}[v^{\pm 1}, q^{\pm 1}]$.*

Proof. It follows from the Theorem 3.2.3 that the matrix α used in (3.2.27) is invertible in $\mathbb{Z}[v^{\pm 1}, q^{\pm 1}]$. $\square$

3.2.4 Twisted Cherednik representation

We will use notation of Subsection 3.2.2. Let us now assume that $d = 1$, $n_{tw} = n'$. In this case, operators $\mathbf{A}_i$ increase grading by 1.

For any $\mathcal{H}_N$-module M denote by $\rho_M \colon \mathcal{H}_N \to \mathrm{End}_{\mathbb{C}}(M)$ the corresponding homomorphism.

Definition 3.2.1. For any $\mathcal{H}_N$-module M and $\tau \in \widetilde{SL}(2, \mathbb{Z})$ let us define the representation M^τ as follows. M and M^τ are the same vector space with different actions, namely $\rho_{M^\tau} = \rho_M \circ \tau^{-1}$.

We will refer to M^τ as a *twisted representation.*

Theorem 3.2.4. *The $\mathbf{C}^{(n,n')}_{u_0,\dots,u_{n-1}}$ is isomorphic to twisted Cherednik representation $\mathbf{C}^\sigma_u$ for σ as in* (3.2.11), (3.2.12) *and $u = u_0 \dots u_{n-1} q^{1-n}$.*

Proof. Let $H^{\mathbf{B}}$ be a copy of affine Hecke algebra generated by T_i and $\mathbf{B}_i$. Twisted Cherednik representation $\mathbf{C}^\sigma_u$ can be interpreted as a representation $\mathcal{H}_N$ induced from one-dimensional representation of $H^{\mathbf{B}}$. As a vector space, $\mathbf{C}^\sigma_u$ is isomorphic to space of Laurent polynomials $\mathbb{C}[\mathbf{A}_1^{\pm 1}, \dots \mathbf{A}_n^{\pm 1}]$.

The vector $e_{(0)^N} \in \mathbf{C}^{(n,n')}_{u_0,\dots,u_{n-1}}$ is eigenvector for $T_1, \cdots T_{N-1}$. Moreover, due to Theorem 3.2.2, vector $e_{(0)^N}$ is an eigenvector for $\mathbf{B}_1, \dots, \mathbf{B}_N$

$$\mathbf{B}_i e_{(0)^N} = u_0 \dots u_{n-1} q^{1-n} v^{2i-1-N} e_{(0)^N}. \tag{3.2.28}$$

Comparing (3.1.18) with (3.2.28), we see that there is a homomrphism $\psi \colon \mathbf{C}^\sigma_u \to \mathbf{C}^{(n,n')}_{u_0,\dots,u_{n-1}}$ determined by $\psi(1) = e_{(0)^N}$.

The twisted Cherednik representation $\mathbf{C}^\sigma_u$ has a basis A_λ. On the other hand, it follows from Theorem 3.2.3 that their images form a basis in $e_{(0)^N} \in \mathbf{C}^{(n,n')}_{u_0,\dots,u_{n-1}}$. Hence the map ψ an isomorphism. $\square$

Remark 3.2.1. There is another way to finish the proof withour using Theorem 3.2.3. Namely, since Cherednik representation is irreducible the map ψ is injective. And it remains ot show that $\mathbf{C}^{(n,n')}_{u_0,\ldots,u_{n-1}}$ is generated from $e_{(0)^N}$ by the action of $T_1.\ldots T_{N-1}, Y_1^{\pm 1}, \ldots Y_N^{\pm 1}, \pi$.

As a corollary we notice, that the representation $\mathbf{C}^{(n,n')}_{u_0,\ldots,u_{n-1}}$ depends (up to an isomorphism) only on the product $u_0 \ldots u_{n-1}$. For example we can take $u_i = u^{\frac{1}{n}} q^{\frac{n+2i-1}{2n}}$. Let us denote

$$\mathbf{C}^{(n,n')}_{u} = \mathbf{C}^{(n,n')}_{u_0,\ldots,u_{n-1}} \qquad \text{for } u_i = u^{\frac{1}{n}} q^{\frac{n+2i-1}{2n}} \tag{3.2.29}$$

3.3 Toroidal algebra

In this section we recall presentations and certain properties of quantum toroidal $\mathfrak{gl}_1$ algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$. In particular, we describe its connection with Double Affine Hecke algebra $\mathcal{H}_N$. The section contains no new results.

Presentations The algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ is an algebra depending on parameters q_1 and q_2. Let us introduce parameter q_3 such that $q_1q_2q_3 = 1$. The algebra has a presentation via generators $P_{a,b}$ for $(a,b) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ and central elements c, c'. We will not need explicit form of the relations, see [BS12b, Def. 6.4] for the reference, relation between our generators and generators in *loc. cit.* is $P_{a,b} = (1-q_1^d)u_{a,b}$, where $d = \gcd(a,b)$.

Proposition 3.3.1 ([BS12b]). *Group $\widetilde{SL}(2,\mathbb{Z})$ acts on $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ via automorphisms.*

Let us consider an element $\tau \in \widetilde{SL}(2,\mathbb{Z})$ such that under the projection (3.1.7) it is mapped

$$\tau \mapsto \begin{pmatrix} m' & m \\ n' & n \end{pmatrix} \tag{3.3.1}$$

Let $\widetilde{SL}(2,\mathbb{R})$ be universal covering of $SL(2,\mathbb{R})$. The group $\widetilde{SL}(2,\mathbb{Z})$ can be interpreted as the preimage of $SL(2,\mathbb{Z})$ in $\widetilde{SL}(2,\mathbb{R})$. Hence we can think about the element $\tau \in \widetilde{SL}(2,\mathbb{Z})$ as a path γ in $SL(2,\mathbb{R})$ from identity matrix to the matrix (3.3.1). The path γ induces a path $\gamma(a,b)$ in $\mathbb{R}^2\backslash\{0,0\}$ by action on (a,b). The intersection number of $\gamma(a,b)$ and the line $a=0$ is called *winding number* $n_\tau(a,b)$[1].

Then the action of τ is given by the following formulas

$$\tau(c) = c^n \left(c'\right)^m \qquad\qquad \tau(c') = c^{n'} \left(c'\right)^{m'} \tag{3.3.2}$$

$$\tau\left(P_{a,b}\right) = \left(\left(c'\right)^{m'a+mb} c^{n'a+nb}\right)^{n_\tau(a,b)} P_{m'a+mb,n'a+nb}, \tag{3.3.3}$$

Chevalley presentation The algebra has another presentation, the equivalence between this two was shown in [Sch12]. The generators are $P_{1,b}$, $P_{0,b}$, $P_{-1,b}$, and central elements c, c'. To describe the relations let us introduce currents

$$E(z) = \sum_{b\in\mathbb{Z}} P_{1,b} z^{-b}, \qquad\qquad F(z) = \sum_{b\in\mathbb{Z}} P_{-1,b} z^{-b}. \tag{3.3.4}$$

Define

$$\sum_{k>0} \theta_{\pm k} z^{-k} = \exp\left(\sum_{k} \frac{(1-q_2^k)(1-q_3^k)}{k} P_{0,\pm k} z^{-k}\right) \tag{3.3.5}$$

[1]if path $\gamma(a,b)$ goes clockwise then (a,b) is included but $\tau(a,b)$ is not included into the path $\gamma(a,b)$. For the conterclockwise path $\gamma(a,b)$ the convention is opposite. See the recent papet [BHM+21, Sec 3.2.1] for the treatment of this group action, bit conventions in *loc. cit.* differ from ours

For $k \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}$

$$[P_{0,k}, P_{0,l}] = k\frac{(1-q_1^k)(c^k - c^{-k})}{(1-q_2^k)(1-q_3^k)}\delta_{k+l,0} \tag{3.3.6a}$$

$$[P_{0,k}, P_{1,b}] = c^{-k}(q_1^k - 1)P_{1,b+k} \qquad [P_{0,-k}, P_{1,b}] = (1-q_1^k)P_{1,b-k} \tag{3.3.6b}$$

$$[P_{0,k}, P_{-1,b}] = (1-q_1^k)P_{-1,b+k} \qquad [P_{0,-k}, P_{-1,b}] = (q_1^k - 1)c^k P_{-1,b-k} \tag{3.3.6c}$$

$$(z-q_1w)(z-q_2w)(z-q_3w)E(z)E(w) = -(w-q_1z)(w-q_2z)(w-q_3z)E(w)E(z) \tag{3.3.6d}$$

$$(z-q_1w)(z-q_2w)(z-q_3w)F(z)F(w) = -(w-q_1z)(w-q_2z)(w-q_3z)F(w)F(z) \tag{3.3.6e}$$

For $a+b>0$

$$[P_{1,a}, P_{-1,b}] = \frac{(1-q_1)c^a c'}{(1-q_2)(1-q_3)}\theta_{a+b} \qquad [P_{1,-a}, P_{-1,-b}] = -\frac{(1-q_1)c^{-b}(c')^{-1}}{(1-q_2)(1-q_3)}\theta_{-a-b} \tag{3.3.6f}$$

For $a \in \mathbb{Z}$

$$[P_{1,a}, P_{-1,-a}] = \frac{(1-q_1)\left(c^a c' - c^{-a}(c')^{-1}\right)}{(1-q_2)(1-q_3)} \tag{3.3.6g}$$

$$[P_{1,a}, [P_{1,a-1}, P_{1,a+1}]] = 0 \tag{3.3.6h}$$

$$[P_{-1,a}, [P_{-1,a-1}, P_{-1,a+1}]] = 0 \tag{3.3.6i}$$

Definition 3.3.1. Algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ is an algebra generated by $P_{1,b}$, $P_{0,b}$, $P_{-1,b}$, and central elements c, c' with the relations (3.3.6a)–(3.3.6i).

Definition 3.3.2. Algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ is an algebra generated by $P_{1,b}$, $P_{0,b}$ and a central element c with the relations (3.3.6a), (3.3.6b), (3.3.6d), (3.3.6h).

Definition 3.3.3. Algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^-$ is an algebra generated by $P_{-1,b}$, $P_{0,b}$ and a central element c with the relations (3.3.6a), (3.3.6c), (3.3.6e), (3.3.6i).

Proposition 3.3.2. *Algebras $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ and $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^-$ are subalgebras of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$.*

Connection with spherical DAHA Denote

$$[i]_v^{\pm} = \frac{v^{\pm 2i} - 1}{v^{\pm 2} - 1} \qquad [k]_v = \frac{v^i - v^{-i}}{v - v^{-1}} \tag{3.3.7}$$

$$[k]!_v^{\pm} = [1]_v^{\pm} \dots [k]_v^{\pm} \qquad [k]!_v = [1]_v \dots [k]_v \tag{3.3.8}$$

In this paragraph we will need to consider Double Affine Hecke Algebras for different parameters q and v, therefore we will write $\mathcal{H}_N(q,v)$. Let $\mathbf{S}_+$ and $\mathbf{S}_-$ be the symmetrizer and anti-symmetrizer in finite Hecke algebra

$$\mathbf{S}_+ = \frac{1}{[N]!_v^+}\sum v^{l(\sigma)}T_\sigma \qquad \mathbf{S}_- = \frac{1}{[N]!_v^-}\sum (-v)^{-l(\sigma)}T_\sigma \tag{3.3.9}$$

The basic property of $\mathbf{S}_\pm$ is that for $i = 1, \dots, N-1$

$$T_i\mathbf{S}_+ = \mathbf{S}_+T_i = v\mathbf{S}_+ \qquad T_i\mathbf{S}_- = \mathbf{S}_-T_i = -v^{-1}\mathbf{S}_- \tag{3.3.10}$$

Let $S_\pm\mathcal{H}_N(q,v) = \mathbf{S}_\pm\mathcal{H}_N(q,v)\mathbf{S}_\pm$ be the corresponding spherical DAHA.

Proposition 3.3.3. *There is algebra isomorphism $S_-\mathcal{H}_N(q,v) \cong S_+\mathcal{H}_N(q,-v^{-1})$.*

Proof. The relations (3.1.1)–(3.1.4) imply that there is an isomorphism $\mu\colon \mathcal{H}_N(q,v) \xrightarrow{\sim} \mathcal{H}_N(q,-v^{-1})$ defined by $\mu(T_i)=T_i$, $\mu(Y_i)=Y_i$, $\mu(\pi)=\pi$. To finish the proof we note that $\mu(\mathbf{S}_-)=\mathbf{S}_+$. $\square$

Theorem 3.3.1 ([SV11]). *The following formulas defines a surjection of the algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ to $S_+\mathcal{H}_N(q,v)$ for $q_1=q$, $q_2=v^2$*

$$P^{(N)}_{0,k} =\mathbf{S}_+(Y_1^k+\cdots+Y_N^k)\mathbf{S}_+ \qquad P^{(N)}_{0,-k} =q^k\mathbf{S}_+(Y_1^{-k}+\cdots+Y_N^{-k})\mathbf{S}_+ \tag{3.3.11a}$$
$$P^{(N)}_{k,0} =q^k\mathbf{S}_+(X_1^k+\cdots+X_N^k)\mathbf{S}_+ \qquad P^{(N)}_{-k,0} =\mathbf{S}_+(X_1^{-k}+\cdots+X_N^{-k})\mathbf{S}_+ \tag{3.3.11b}$$
$$P^{(N)}_{1,b} =q[N]_v^-\mathbf{S}_+X_1Y_1^b\mathbf{S}_+ \qquad P^{(N)}_{-1,b} =[N]_v^+\mathbf{S}_+Y_1^bX_1^{-1}\mathbf{S}_+ \tag{3.3.11c}$$

Here $k\in\mathbb{Z}_{>0}$ and $b\in\mathbb{Z}$ and the image of $P_{a,b}$ is denoted by $P^{(N)}_{a,b}$.

Remark 3.3.1. In [SV11] the authors prove that quotient of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ by relations $c=c'=1$ is a projective limit of $S\mathcal{H}_N$. This deep result is one of the motivations for our work. Though, formally speaking, we will not use it.

Corollary 3.3.4. *$U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ surjects to $S_-\mathcal{H}_N(q,v)$ for $q_1=q$, $q_2=v^{-2}$. Moreover*

$$P^{(N)}_{0,k} =\mathbf{S}_-(Y_1^k+\cdots+Y_N^k)\mathbf{S}_- \qquad P^{(N)}_{0,-k} =q^k\mathbf{S}_-(Y_1^{-k}+\cdots+Y_N^{-k})\mathbf{S}_- \tag{3.3.12a}$$
$$P^{(N)}_{k,0} =q^k\mathbf{S}_-(X_1^k+\cdots+X_N^k)\mathbf{S}_- \qquad P^{(N)}_{-k,0} =\mathbf{S}_-(X_1^{-k}+\cdots+X_N^{-k})\mathbf{S}_- \tag{3.3.12b}$$
$$P^{(N)}_{1,b} =q[N]_v^+\mathbf{S}_-X_1Y_1^b\mathbf{S}_- \qquad P^{(N)}_{-1,b} =[N]_v^-\mathbf{S}_-Y_1^bX_1^{-1}\mathbf{S}_- \tag{3.3.12c}$$

3.4 Deformed exterior power

The algebra $S_-\mathcal{H}_N(q,v)$ acts on the subspace $\mathbf{S}_-\mathbf{C}_u^{(n,n')}=\mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N}$. The space $\mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N}$ was considered in [KMS95, LT00]. In *loc. cit.*, the authors considered only affine Hecke algebra action on $\left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N}$, but not DAHA. In this section we will recall and extend their results. Spherical DAHA will be considered in the subsequent sections.

3.4.1 Finite v-wedge

The v-deformd exterior power can be defined as a subspace $\mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N}$. On the other hand it can be identified with the quotient space via tautological projection

$$\mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N} \xrightarrow{\sim} \left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N}\Big/\sum_i \mathrm{Im}(T_i+v^{-1}) \tag{3.4.1}$$

The inverse map is induced by $\mathbf{S}_-$. We will use both interpretations as a subspace and as a quotient. Denote by $e_{i_1}\wedge\cdots\wedge e_{i_N}=\mathbf{S}_-\left(e_{i_1}\otimes\cdots\otimes e_{i_N}\right)$.

Lemma 3.4.1 ([KMS95, eqs. (41), (42)]). *Let $l=m+nk+j$ for $k\geq0$ and $0\leq j<n$. Then*

$$e_l\wedge e_m = -\,e_m\wedge e_l \qquad \textit{for } j=0 \tag{3.4.2a}$$
$$e_l\wedge e_m = -\,ve_m\wedge e_l \qquad \textit{for } k=0 \tag{3.4.2b}$$
$$e_l\wedge e_m = -\,ve_m\wedge e_l - e_{l-nk}\wedge e_{m+nk} - ve_{m+nk}\wedge e_{l-nk} \qquad \textit{otherwise} \tag{3.4.2c}$$

The above identities can be used for a vectors of the form $e_{i_1}\wedge\cdots\wedge e_l\wedge e_m\wedge\cdots\wedge e_{i_N}$.

Proposition 3.4.1 ([KMS95, Prop 1.3]). *Vectors $e_{i_1}\wedge\cdots\wedge e_{i_N}$ for $i_1<i_2<\cdots<i_N$ form a basis of $\mathbf{S}_-(\mathbb{C}^n[Y^{\pm1}])^{\otimes N}$.*

Lemma 3.4.2 ([LT00, Lemma 7.6]). *Let $k_1,\ldots,k_N$ be integers such that $\sum_{i=1}^N(i-m-k_i)<N$ and all $k_i<N-m$ for certain $m\in\mathbb{Z}$. Then $e_{k_1}\wedge\cdots\wedge e_{k_N}=0$.*

Vertex operators Considering w as an element of subspace we will write it as $w = \sum_{j_1,\dots,j_N} y_{j_1,\dots,j_N} e_{j_1} \otimes \cdots \otimes e_{j_N} \subset \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$. Considering w as a an element of the quotient we will write $w = \sum_{i_1,\dots,i_N} x_{i_1,\dots,i_N} e_{i_1} \wedge \cdots \wedge e_{i_N}$.

Let us define (modes of) vertex operators $\Phi_k, \Psi_k \colon \mathbf{S}_-(\mathbb{C}^n[Y^{\pm 1}])^{\otimes N} \to \mathbf{S}_-(\mathbb{C}^n[Y^{\pm 1}])^{\otimes (N+1)}$ by the formula

$$\Phi_k(w) = \sum_{j_1,\dots,j_N} x_{j_1,\dots,j_N} e_k \wedge e_{j_1} \wedge \cdots \wedge e_{j_N}, \quad \Psi_k(w) = \sum_{j_1,\dots,j_N} x_{j_1,\dots,j_N} e_{j_1} \wedge \cdots \wedge e_{j_N} \wedge e_k, \tag{3.4.3}$$

here w is considered as an element of the quotient. Note also that vertex operators Φ_k, Ψ_k can also be defined in terms of subspace.

$$\begin{aligned}\Phi_k(w) &= \mathbf{S}_-^{(N+1)}\Big(\sum_{i_1,\dots,i_N} x_{i_1,\dots,i_N} e_k \otimes e_{i_1} \otimes \cdots \otimes e_{i_N}\Big) \\ &= \mathbf{S}_-^{(N+1)}\mathbf{S}_-^{(N)}\Big(\sum_{i_1,\dots,i_N} x_{i_1,\dots,i_N} e_k \otimes e_{i_1} \otimes \cdots \otimes e_{i_N}\Big) = \mathbf{S}_-^{(N+1)} \sum_{j_1,\dots,j_N} y_{j_1,\dots,j_N}\, e_k \otimes e_{j_1} \otimes \cdots \otimes e_{j_N},\end{aligned}$$

where $\mathbf{S}_-^{(N)}$ denotes anti-symmetrizer in $\mathcal{H}_N$.

Define (modes of) dual vertex operators $\Phi_k^*, \Psi_k^* \colon \mathbf{S}_-(\mathbb{C}^n[Y^{\pm 1}])^{\otimes N} \to \mathbf{S}_-(\mathbb{C}^n[Y^{\pm 1}])^{\otimes (N-1)}$ by the formula

$$\Phi_k^*(w) = \sum_{j_2,\dots,j_N} y_{-k,j_2\dots,j_N}\, e_{j_2} \otimes \cdots \otimes e_{j_N}, \quad \Psi_k^*(w) = \sum_{j_1,\dots,j_{N-1}} y_{j_1,\dots,j_{N-1},-k}\, e_{j_1} \otimes \cdots \otimes e_{j_{N-1}}. \tag{3.4.4}$$

Here w is considered as an element of the subspace. It is easy to see that

$$(T_i + v^{-1}) \sum_{j_2,\dots,j_N} y_{-k,j_2\dots,j_N}\, e_{j_2} \otimes \cdots \otimes e_{j_N} = (T_i + v^{-1}) \sum_{j_1,\dots,j_{N-1}} y_{j_1,\dots,j_{N-1},-k}\, e_{j_1} \otimes \cdots \otimes e_{j_{N-1}} = 0$$

for any $1 \le i \le N-2$. Hence the image of Φ_k^* and Ψ_k^* indeed belongs to $\mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N-1}$.

Consider operators

$$b_j = Y_1^j + \cdots + Y_N^j. \tag{3.4.5}$$

Lemma 3.4.3. *The following relaitons hold for $k \in \mathbb{Z}$ and $j \in \mathbb{Z}_{\neq 0}$*

$$[b_j, \Phi_k] = \Phi_{k+nj} \qquad [b_j, \Psi_k] = \Psi_{k+nj} \qquad [b_j, \Phi_k^*] = -\Phi_{k+nj}^* \qquad [b_j, \Psi_k^*] = -\Psi_{k+nj}^* \tag{3.4.6}$$

Proof. Follows directly from the following formulas

$$b_k \sum_{i_1,\dots,i_N} x_{i_1,\dots,i_N} e_{i_1} \wedge \cdots \wedge e_{i_N} = \sum_{i_1,\dots,i_N} \sum_{r=1}^{N} x_{i_1,\dots,i_N} e_{i_1} \wedge \cdots \wedge e_{i_r+kn} \wedge \cdots \wedge e_{i_N}$$

$$b_k \sum_{j_1,\dots,j_N} y_{j_1,\dots,j_N} e_{j_1} \otimes \cdots \otimes e_{j_N} = \sum_{j_1,\dots,j_N} \sum_{r=1}^{N} y_{j_1,\dots,j_N} e_{j_1} \otimes \cdots \otimes e_{j_r+kn} \otimes \cdots \otimes e_{j_N}.$$

□

Involution In order to prove certain propertirs of vertex operators we will need bar involution. This is anilinear map $\bar{v} = v^{-1}$ and its action on $\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$ given by the formula [LT00, Prop. 5.5]

$$\overline{e_{j_1} \otimes \cdots \otimes e_{j_N}} = v^{N(N-1)/2 - \mathrm{b_j}} T_{w_0} e_{j_N} \otimes \cdots \otimes e_{i_1}. \tag{3.4.7}$$

Here $b_{\mathbf{j}}$ is the number of pairs (j_r, j_s) such that $j_r \not\equiv j_s \bmod n$, w_0 is the longest element in Weyl group and T_{w_0} is the corresponding element of the Hecke algebra. This is an *ad hoc* definition, see [LT00] for more details.

Bar involution preserves the subspace $\mathbf{S}_-\big(\mathbb{C}^n[Y^{\pm 1}]\big)^{\otimes N}$ and acts by the formula [LT00, Prop. 5.9]

$$\overline{e_{i_1} \wedge \cdots \wedge e_{i_N}} = (-1)^{N(N-1)/2} v^{-b_{\mathbf{i}}} e_{i_N} \wedge \cdots \wedge e_{i_1} \tag{3.4.8}$$

Lemma 3.4.4. *The following holds*

$$(-1)^N \overline{v^{-a_{\mathbf{i}}(k)} \Psi_k \overline{e_{i_1} \wedge \cdots \wedge e_{i_N}}} = \Phi_k e_{i_1} \wedge \cdots \wedge e_{i_N} \tag{3.4.9}$$

$$(-1)^{N-1} \overline{v^{a_{\mathbf{i}}(k)} \Psi_k^* \overline{e_{i_1} \wedge \cdots \wedge e_{i_N}}} = \Phi_k^* e_{i_1} \wedge \cdots \wedge e_{i_N}. \tag{3.4.10}$$

here $a_{\mathbf{i}}(k)$ is the number of r such that $i_r \not\equiv k \bmod n$.

Proof. For the first relation we use (3.4.8) we get

$$\begin{aligned}\overline{\Psi_k \overline{e_{i_1} \wedge \cdots \wedge e_{i_N}}} &= \overline{\Psi_k (-1)^{N(N-1)/2} v^{-b_{\mathbf{i}}} e_{i_N} \wedge \cdots \wedge e_{i_1}} = (-1)^{N(N-1)/2} v^{b_{\mathbf{i}}} \overline{e_{i_N} \wedge \cdots \wedge e_{i_1} \wedge e_k} \\ &= (-1)^N v^{-a_{\mathbf{i}}} e_k \wedge e_{i_1} \wedge \cdots \wedge e_{i_N} = (-1)^N v^{-a_{\mathbf{i}}} \Psi_k e_{i_1} \wedge \cdots \wedge e_{i_N}.\end{aligned}$$

For the second relation we use notation $e_{i_1} \wedge \cdots \wedge e_{i_N} = \sum_{\mathbf{j}} y_{\mathbf{j}} e_{j_1} \otimes \cdots \otimes e_{j_N}$. Then

$$\overline{e_{i_1} \wedge \cdots \wedge e_{i_N}} = (-v)^{-N(N-1)/2} T_{w_0}^{-1} \overline{e_{i_1} \wedge \cdots \wedge e_{i_N}} = (-1)^{N(N-1)/2} v^{-b_{\mathbf{i}}} \sum_{\mathbf{j}} \bar{y}_{\mathbf{j}}\, e_{j_N} \otimes \cdots \otimes e_{j_1}.$$

Here we used relations (3.4.7) and $b_{\mathbf{i}} = b_{\mathbf{j}}$. Therefore using this relation twice we obtain

$$\begin{aligned}\overline{\Psi_k^* \overline{e_{i_1} \wedge \cdots \wedge e_{i_N}}} &= \sum_{\mathbf{j}} (-1)^{N(N-1)/2} v^{b_{\mathbf{i}}} y_{\mathbf{j}} \overline{\Psi_k^* e_{j_N} \otimes \cdots \otimes e_{j_1}} \\ &= \sum_{j_2,\ldots,j_N} (-1)^{N(N-1)/2} v^{b_{\mathbf{i}}} y_{-k,j_2,\ldots,j_N} \overline{e_{j_N} \otimes \cdots \otimes e_{j_2}} \\ &= (-1)^{(N-1)} v^{a_{\mathbf{i}}} \overline{\overline{\Phi_k^* e_{i_1} \wedge \cdots \wedge e_{i_N}}} = (-1)^{(N-1)} v^{a_{\mathbf{i}}} \Phi_k^* e_{i_1} \wedge \cdots \wedge e_{i_N}\end{aligned}$$

□

3.4.2 The limit

Let us consider an inductive system of vector spaces

$$\mathbf{S}_-\mathbb{C}^n[Y^{\pm 1}] \xrightarrow{\varphi_{2,1}^{(m)}} \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes 2} \xrightarrow{\varphi_{3,2}^{(m)}} \ldots \xrightarrow{\varphi_{N,N-1}^{(m)}} \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \xrightarrow{\varphi_{N+1,N}^{(m)}} \ldots \tag{3.4.11}$$

with the maps $\varphi_{N+1,N}^{(m)}(w) = w \wedge e_{N-m}$. Denote the injective limit by $\Lambda_{v,m}^{\infty/2}\big(\mathbb{C}^n[Y^{\pm 1}]\big)$. Also let us define maps $\varphi_{R,N}^{(m)}(w) = w \wedge e_{N-m} \wedge e_{N+1-m} \wedge \cdots \wedge e_{R-1-m}$ for $R > N$.

By $\varphi_N^{(m)} = \varphi_{\infty,N}^{(m)} \colon \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \to \Lambda_{v,m}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$ we denote the canonical map.

Definition 3.4.1. Action of (a sequence of) operators $A^{(N)} \colon \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \to \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes(N+\delta)}$ *stabilizes* if for any $w \in \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes k}$ there is M such that for any $N > M$ we have

$$\varphi_{N+n+\delta,N+\delta}^{(m+\delta)} \circ A^{(N)} \varphi_{N,k}^{(m)}(w) = A^{(N+n)} \circ \varphi_{N+n,k}^{(m)}(w) \tag{3.4.12}$$

Note that if action of $A^{(N)}$ stabilizes, then it induces an operator $\hat{A} \colon \Lambda_{v,m}^{\infty/2}\big(\mathbb{C}^n[Y^{\pm 1}]\big) \to \Lambda_{v,m+\delta}^{\infty/2}\big(\mathbb{C}^n[Y^{\pm 1}]\big)$. Actually, the operator $\hat{A}$ depends on residue of N modulo n. We will omit this dependence in our notation.

Proposition 3.4.2. *Let $A_1^{(N)}$ stabilizes and $A_2^{(N)}$ stabilizes. Then the composition $A_1^{(N)}A_2^{(N)}$ stabilizes and the induced operator equals the composition of induced operators $\hat{A}_1\hat{A}_2$.*

Proposition 3.4.3. *[LT00, Sec. 7.6] Action of bar involution stabilizes.*

For any partition λ with $l(\lambda) \leq N$ consider a vector

$$|\lambda\rangle = e_{-\lambda_1-m} \wedge e_{-\lambda_2+1-m} \wedge \cdots \wedge e_{-\lambda_r+r-1-m} \wedge e_{r-m} \wedge \cdots \wedge e_{N-1-m} \tag{3.4.13}$$

We will denote this vector as $|\lambda\rangle_{N,m}$ if these indices were not clear from the context. We will write $|\lambda\rangle_{\infty,m} = \varphi_{\infty,N}^{(m)}|\lambda\rangle_{N,m}$.

Lemma 3.4.5. *For $m - k + |\lambda| < N$ we have*

$$[N]_v^+ \left(\Phi_k^*|\lambda\rangle\right) \wedge e_{N-m} = [N+1]_v^+ \Phi_k^* \Big(|\lambda\rangle \wedge e_{N-m}\Big) \tag{3.4.14}$$

Proof. Let us introduce notation $|\lambda\rangle = \sum_{j_1,\ldots,j_N} y_{j_1,\ldots,j_N} e_{j_1} \otimes \cdots \otimes e_{j_N}$. Then we have

$$\begin{aligned}\text{LHS of (3.4.14)} = [N]_v^+ (\textstyle\sum_{j_2,\ldots,j_N} y_{-k,\ldots,j_N} e_{j_2} \otimes \cdots \otimes e_{j_N}) \wedge e_{N-m} \\ = [N]_v^+ \textstyle\sum_{j_2,\ldots,j_N} y_{-k,\ldots,j_N} e_{j_2} \wedge \cdots \wedge e_{j_N} \wedge e_{N-m}\end{aligned} \tag{3.4.15}$$

In order to compute RHS of (3.4.14) we will use factorization formula

$$\mathbf{S}_-^{(N+1)} = \frac{1}{[N+1]_v^+} \left(\sum_{p=1}^{N+1} (-v)^{p+N-1} T_p \cdot \cdots \cdot T_N\right) \mathbf{S}_-^{(N)}$$

Using this we obtain that

$$|\lambda\rangle \wedge e_{N-m} = \frac{1}{[N+1]_v^+} \left(\sum_{p=1}^{N+1} (-v)^{p+N-1} T_p \cdot \cdots \cdot T_N\right) \left(\sum_{\mathbf{j}} y_{\mathbf{j}}\, e_{j_1} \otimes \cdots \otimes e_{j_N} \otimes e_{N-m}\right)$$

Here and below we use multi-index notation $\mathbf{j} = (j_1, \ldots, j_N)$. For the action of $T_i \cdot \cdots \cdot T_N$ we will use formulas (3.2.6)–(3.2.9). Informally, the formulas say that under the action of T, the vectors either remains the same, or permute, or *approach* to each other. Hence we get

$$\begin{aligned}
[N+1]_v^+ |\lambda\rangle \wedge e_{N-m} &= \Big(\sum_{\mathbf{j}} \Big(\sum_{p=1}^{N} (-v)^{p+N-1} T_p \cdot \ldots T_{N-1} y_{\mathbf{j}} \\
&\quad \big(v^{\delta_{j_N \equiv N-m}} e_{j_1} \otimes \ldots \otimes e_{j_{N-1}} \otimes e_{N-m} \otimes e_{j_N} + (v - v^{-1}) e_{j_1} \otimes \ldots \otimes e_{j_N} \otimes e_{N-m}\big) \\
&\quad + (-v)^{2N} y_{\mathbf{j}}\, e_{j_1} \otimes \ldots \otimes e_{j_N} \otimes e_{N-m}\Big)\Big) + \text{lower terms} \\
&= \Big(\sum_{\mathbf{j}} \Big(\sum_{p=1}^{N} (-v)^{p+N-1} v^{\delta_{j_N \equiv N-m}} T_p \cdot \ldots T_{N-1} y_{\mathbf{j}}\, e_{j_1} \otimes \ldots \otimes e_{j_{N-1}} \otimes e_{N-m} \otimes e_{j_N} + \\
&\quad \big((-v)^{2N} + \sum_{p=1}^{N} (v - v^{-1})(-v)^{p+N-1}(-v)^{p-N}\big) y_{\mathbf{j}}\, e_{j_1} \otimes \ldots \otimes e_{j_N} \otimes e_{N-m}\Big)\Big) + \text{lower terms} \\
&= \cdots = \sum_{q=1}^{N+1} \sum_{\mathbf{j}} v^{\sum_{r=q}^{N} \delta_{j_r \equiv N-m}} \Big((-v)^{q+N-1} + (v - v^{-1}) \sum_{p=1}^{q-1} (-v)^{p+N-1}(-v)^{p-q+1}\Big) \\
&\quad y_{\mathbf{j}}\, e_{j_1} \otimes \ldots \otimes e_{j_{q-1}} \otimes e_{N-m} \otimes e_{j_q} \otimes \ldots \otimes e_{j_N} + \text{lower terms} \\
&= \sum_{q=1}^{N+1} \sum_{\mathbf{j}} v^{\sum_{r=q}^{N} \delta_{j_r \equiv N-m}} (-v)^{N-q+1} y_{\mathbf{j}}\, e_{j_1} \otimes \ldots \otimes e_{j_{q-1}} \otimes e_{N-m} \otimes e_{j_q} \otimes \ldots \otimes e_{j_N} + \text{lower terms.}
\end{aligned}$$

Here *lower terms* stands for linear combination of terms $e_{l_1} \otimes \cdots \otimes e_{l_{N+1}}$ where $l_i < N - m$, $\forall i$. In the computation we used (3.2.6), (3.2.8) and relation $T_p \sum_{\mathbf{j}} y_{\mathbf{j}} e_{j_1} \otimes \ldots \otimes e_{j_{q-1}} \otimes e_{N-m} \otimes e_{j_q} \otimes \ldots \otimes e_{j_N} = (-v)^{-1} \sum_{\mathbf{j}} y_{\mathbf{j}} e_{j_1} \otimes \ldots \otimes e_{j_{q-1}} \otimes e_{N-m} \otimes e_{j_q} \otimes \ldots \otimes e_{j_N}$ for $p < q$.

In order to compute RHS of (3.4.14) we apply $\mathbf{S}_-^{(N)}\Phi_k^*$. Due to Lemma 3.4.2 the *lower terms* vanish after $\mathbf{S}_-^{(N)}$. Hence we get

$$\begin{aligned}
&\text{RHS of (3.4.14)}\\
&= \sum_{j_2,\dots,j_N} \sum_{q=2}^{N+1} v^{\sum_{r=q}^N \delta_{j_r \equiv N-m}} (-v)^{N-q+1} y_{-k,j_2,\dots,j_N} e_{j_2} \wedge \dots \wedge e_{j_{q-1}} \wedge e_{N-m} \wedge e_{j_q} \wedge \dots \wedge e_{j_N}\\
&= \sum_{j_2,\dots,j_N} \sum_{q=2}^{N+1} (-v)^{2N+2-2q} y_{-k,j_2,\dots,j_N} e_{j_2} \wedge \dots \wedge e_{j_N} \wedge e_{N-m}\\
&= [N]_v^+ \sum_{j_2,\dots,j_N} y_{-k,j_2,\dots,j_N} e_{j_2} \wedge \dots \wedge e_{j_N} \wedge e_{N-m} \qquad (3.4.16)
\end{aligned}$$

where we used Lemma 3.4.1 to permute e_{N-m} to the right and Lemma 3.4.2 to cancel out additional lower terms. Comparing the formulas (3.4.15) and (3.4.16) we get the result. □

Proposition 3.4.4. *Action of Φ_k and $\tilde{\Phi}_k^* = [N]_v^+ \Phi_k^*$ stabilize.*

Proof. Lemma 3.4.1 imply, that action of Φ_k stabilizes for $N - m > k$. Lemma 3.4.5 imply that $\tilde{\Phi}_k^*$ stabilizes. □

Remark 3.4.1. We used Lemma 3.4.5 in the proof above. But below in Section 3.5 we will need a refinement of this result. Let us introduce the following notation

$$\left(\tilde{\Phi}_k^* |\lambda\rangle\right) = \sum_\mu x_\mu e_{-\mu_1+1-m} \wedge \dots \wedge e_{-\mu_{N-1}+N-m-1}, \tag{3.4.17}$$

$$\left(\tilde{\Phi}_k^* \left(|\lambda\rangle \wedge e_{N-m}\right)\right) = \sum_\mu x'_\mu e_{-\mu_1+1-m} \wedge \dots \wedge e_{-\mu_N+N-m}, \tag{3.4.18}$$

Then for $l(\mu) \leq N - 1$ we have $x_\mu = x'_\mu$.

The proof is based on the same computation as proof of Lemma 3.4.5. We redefine *lower terms* as terms containing only e_l with $l < N - m$. The action of $\mathbf{S}_-$ cannot give $e_{-\mu_1+1-m} \wedge \dots \wedge e_{-\mu_N+N-m}$ with $\mu_N = 0$ (i.e. terms containing e_{N-m}).

Proposition 3.4.5. *The following operators stabilize*

$$\tilde{\Psi}_k = (-1)^N v^{-\frac{n-1}{n}N} \Psi_k \qquad \tilde{\Psi}_k^* = (-1)^N [N]_v^- v^{\frac{n-1}{n}N} \Psi_k^* \tag{3.4.19}$$

Proof. Follows from the previous proposition and Lemma 3.4.4. □

Let us denote the induced operators as follows

$$\hat{\Phi}_k^* \colon \Lambda_{v,m}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right) \to \Lambda_{v,m-1}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right) \qquad \hat{\Psi}_k \colon \Lambda_{v,m-1}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right) \to \Lambda_{v,m}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right) \tag{3.4.20}$$

$$\hat{\Psi}_k^* \colon \Lambda_{v,m}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right) \to \Lambda_{v,m-1}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right) \qquad \hat{\Phi}_k \colon \Lambda_{v,m-1}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right) \to \Lambda_{v,m}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right) \tag{3.4.21}$$

Definition 3.4.2. Action of (a sequence of) operators $A^{(N)} \colon \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \to \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes (N+\delta)}$ *weakly stabilizes* if for any $w \in \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes k}$ there is M such that for any $N > M$ we have

$$\varphi_{N+\delta}^{(m+\delta)} \circ A^{(N)} \varphi_{N,k}^{(m)}(w) = \varphi_{N+n+\Delta}^{(m+\delta)} \circ A^{(N+n)} \circ \varphi_{N+n,k}^{(m)}(w) \tag{3.4.22}$$

Proposition 3.4.6 ([KMS95])**.** *Action of operators $b_k = Y_1^k + \dots + Y_N^k$ stabilizes for $k < 0$ and weakly stabilizes for $k > 0$. The induced operators B_k satisfy deformed Heisenberg algebra relation*

$$[B_k, B_l] = k[n]_{v^k}^+ \delta_{k+l,0} \tag{3.4.23}$$

Example 3.4.1. The operators $b_1 = Y_1 + \cdots + Y_N$ do not stabilize. It can be seen from the formula

$$b_1 e_0 \wedge e_1 \wedge \cdots \wedge e_{N-1} = \sum_{k=0}^{n-1} (-v)^k e_0 \wedge e_1 \wedge \cdots \wedge \hat{e}_{N-1-k} \wedge \cdots \wedge e_{N-1} \wedge e_{N-1-k+n} \tag{3.4.24}$$

Definitely, RHS of (3.4.24) does not belong to the image of $\varphi^{(0)}_{N,N-n}$. Though RHS of (3.4.24) belongs to the kernel of $\varphi^{(0)}_N$. Hence $B_1\varphi^{(0)}_N(e_1 \wedge \cdots \wedge e_{N-1}) = 0$. Moreover note that

$$b_{-1}b_1 e_0 \wedge e_1 \wedge \cdots \wedge e_{N-1} = [n]^+_v e_0 \wedge e_1 \wedge \cdots \wedge e_{N-1} + \ker \varphi^{(0)}_N \tag{3.4.25}$$

$$B_{-1}B_1 e_0 \wedge e_1 \wedge \cdots \wedge e_{N-1} \wedge \cdots = 0 \tag{3.4.26}$$

This means that composition of induced operators does not have to be equal to induced operator of the composition (if the second operator weakly stabilizes). Also, this illustrates the fact that $[B_1, B_{-1}] = [n]^+_v$, though $[b_1, b_{-1}] = 0$.

Proposition 3.4.7. *Let $A^{(N)}_1$ weakly stabilizes and $A^{(N)}_2$ stabilizes. Then the composition $A^{(N)}_1 A^{(N)}_2$ weakly stabilizes and the induced operator equals the composition of induced operators $\hat{A}_1\hat{A}_2$.*

Proposition 3.4.8. *The following relaitons hold for all $k \in \mathbb{Z}$ and $j > 0$*

$$[B_{-j}, \hat{\Phi}_k] = \hat{\Phi}_{k-nj}, \quad [B_{-j}, \hat{\Psi}_k] = \hat{\Psi}_{k-nj}, \quad [B_{-j}, \hat{\Phi}^*_k] = -\hat{\Phi}^*_{k-nj}, \quad [B_{-j}, \hat{\Psi}^*_k] = -\hat{\Psi}^*_{k-nj}, \tag{3.4.27}$$

$$[B_j, \hat{\Phi}_k] = \hat{\Phi}_{k+nj}, \quad [B_j, \hat{\Psi}_k] = v^{2j(n-1)}\hat{\Psi}_{k+nj}, \quad [B_j, \hat{\Phi}^*_k] = -v^{2jn}\hat{\Phi}^*_{k+nj}, \quad [B_j, \hat{\Psi}^*_k] = -v^{-2j}\hat{\Psi}^*_{k+nj}. \tag{3.4.28}$$

Proof. Commutation relations (3.4.27) follows from (3.4.6) since B_{-j} stabilizes. Also, one can check that

$$[B_j, \hat{\Phi}_k]\varphi_N(w) = \varphi^{(m)}_N([b_j, \Phi_k]w) = \varphi^{(m)}_N(\Phi_{k+nj}w) = \hat{\Phi}_{k+nj}\varphi^{(m)}_N(w). \tag{3.4.29}$$

For the relation with $\hat{\Psi}_k$ we use Lemma 3.4.4 and [LT00, Prop. 7.8] that $\overline{B_j} = v^{-2j(n-1)}B_j$ for $j > 0$. Hence

$$\begin{aligned} [B_j, \hat{\Psi}_k]|\lambda\rangle &= \overline{(-1)^N[v^{-2j(n-1)}B_j, v^{-\mathsf{a}_\lambda(k)}\hat{\Phi}_k]\overline{|\lambda\rangle}} \\ &= \overline{(-1)^N v^{-2j(n-1)} \times v^{-\mathsf{a}_\lambda(k)}\hat{\Phi}_{k+nj}\overline{|\lambda\rangle}} = v^{2j(n-1)}\hat{\Psi}_{k+nj}|\lambda\rangle, \end{aligned} \tag{3.4.30}$$

here $\mathsf{a}_\lambda(k)$ equals to $a_\lambda(k) - \frac{n-1}{n}N$ for sufficiently large N.[2]

The relation $[B_j, \hat{\Phi}^*_k] = -v^{2jn}\hat{\Phi}^*_{k+nj}$ is proven in Section 3.8.3 (Theorem 3.8.1). Finally,

$$\begin{aligned} [B_j, \hat{\Psi}^*_k]|\lambda\rangle &= \overline{(-1)^{N-1}[v^{-2j(n-1)}B_j, v^{\mathsf{a}_\lambda(k)}\hat{\Phi}^*_k]\overline{|\lambda\rangle}} \\ &= \overline{(-1)^N v^{-2j(n-1)+2jn} \times v^{\mathsf{a}_\lambda(k)}\hat{\Phi}^*_{k+jn}\overline{|\lambda\rangle}} = -v^{-2j}\hat{\Psi}^*_{k+jn}|\lambda\rangle. \end{aligned} \tag{3.4.31}$$

□

3.5 Semi-infinite construction of twisted Fock module I

In this and the next sections, we will provide an explicit construction for the action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ on twisted Fock module $\mathcal{F}^\sigma_u$ (Theorem 3.6.1). This is the central result of the whole paper. Our method is

[2] We use notation $a_\lambda(k) = a_{\mathfrak{i}}(k)$ for $i_j = -\lambda_j + j - 1 - m$, see Lemma 3.4.4

semi-infinite construction. Namely, we will use explicit construction of $\mathbf{C}_u^\sigma \cong \mathbf{C}_u^{(n,n')}$ (Theorem 3.2.1) to derive explicit construction of $\mathcal{F}_u^\sigma$ as a limit $N \to \infty$.

In subsection 3.5.2, we will study the limit $N \to \infty$ for the Chevalley generators (after a rescaling) denoted by $\tilde{P}_{1,b}^{(N)}$, $\tilde{P}_{0,b}^{(N)}$, and $\tilde{P}_{-1,b}^{(N)}$. It turns out that the generators $\tilde{P}_{1,b}^{(N)}$ and $\tilde{P}_{-1,b}^{(N)}$ converge for $|q^{-1}v^2| < 1$ and $|q^{-1}v^2| > 1$ correspondingly. Therefore we can not obtain the action of whole $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ by straightforward limit argument. Though we prove that we do obtain actions of the subalgebras $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ and $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^-$.

We obtain explicit formulas for the limit of Chevalley generators. The formulas allow us to make analytic continuation for general q and v (subsection 3.5.3). Also, we consider the formulas in the case $n = 1$, $n' = 0$ and show that the obtained operators give Fock module of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ (subsection 3.5.4). We will prove for general n and n' that the obtained operators give twisted Fock module of whole $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ in Section 3.6.

3.5.1 Finite case

It follows from the Corollary 3.3.4 that

$$P_{1,b}^{(N)} =(-1)^{N-1}q[N]_v\mathbf{S}_-\pi^{-1}Y_1^b\mathbf{S}_-, \qquad P_{-1,b}^{(N)} =(-1)^{N-1}[N]_v\mathbf{S}_-Y_1^b\pi\mathbf{S}_-. \tag{3.5.1}$$

Recall the representation $\mathbf{C}_u^{(n,n')}$ defined by (3.2.29). Theorem 3.2.1 implies that

$$P_{0,k}^{(N)} =b_k, \qquad P_{0,-k}^{(N)} = q^k b_{-k}, \tag{3.5.2a}$$

$$P_{1,b}^{(N)} =(-1)^{N-1}u^{-\frac{1}{n}}q^{\frac{n+1}{2n}}[N]_v\sum_{k\in\mathbb{Z}}q^{(-k-nb-n')/n}\Psi_{k+n'+nb}\Phi^*_{-k}, \tag{3.5.2b}$$

$$P_{-1,b}^{(N)} =(-1)^{N-1}u^{\frac{1}{n}}q^{\frac{n-1}{2n}}[N]_v\sum_{k\in\mathbb{Z}}q^{k/n}\Phi_{k-n'+nb}\Psi^*_{-k}. \tag{3.5.2c}$$

Note that for each vector $w \in \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N}$ only finitely many terms at RHS of (3.5.2b)-(3.5.2c) have non-zero action.

3.5.2 The limit for the right and left halves

Below we will construct an action $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ on $\Lambda_{v,m}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm1}]\right)$. Analogous results holds for $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^-$. To simplify our notation, we will consider the case $m = 0$, and $u^{-\frac{1}{n}}q^{\frac{n+1}{2n}} = 1$. We will recover these parameters at the end.

Proposition 3.5.1. *The operators $\tilde{P}_{1,b}^{(N)} = v^{\frac{N}{n}}P_{1,b}^{(N)}$ stabilizes for $n' + nb < 0$. The induced operator*

$$\hat{P}_{1,b} = \sum_{k\in\mathbb{Z}}q^{(-k-nb-n')/n}\hat{\Psi}_{k+n'+nb}\hat{\Phi}^*_{-k}. \tag{3.5.3}$$

For any vector $\hat{w} \in \Lambda_{v,0}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm1}]\right)$, only finitely many terms $\hat{\Psi}_{k+n'+nb}\hat{\Phi}^*_{-k}\hat{w}$ are non-zero.

Recall the notation $|\lambda\rangle$ introduces in (3.4.13). We will need the following lemma.

Lemma 3.5.1. $\Psi_{k-\Delta}\Phi^*_{-k}|\lambda\rangle = 0$ *for* $\Delta \geq 0$ *and* $k \geq |\lambda| + \Delta$.

Proof. Introduce notation $|\lambda\rangle = \sum_{\mathbf{j}} y_{\mathbf{j}}e_{j_1} \otimes \cdots \otimes e_{j_N}$. Then

$$\Psi_{k-\Delta}\Phi^*_{-k}\,|\lambda\rangle = \sum_{j_2,\ldots,j_{N+1}} y_{k,\ldots,j_{N+1}}e_{j_2} \wedge \cdots \wedge e_{j_N} \wedge e_{k-\Delta}. \tag{3.5.4}$$

Consider the decomposition with respect to the basis $|\mu\rangle$

$$\sum_{j_2,\dots,j_N} y_{k,\dots,j_{N+1}} e_{j_2}\wedge\dots\wedge e_{j_N}\wedge e_{k-\Delta} = \sum_{\mu_1\geq\dots\geq\mu_n} x_{\mu_1,\dots,\mu_N} e_{-\mu_1}\wedge e_{-\mu_2+1}\dots\wedge e_{-\mu_N+N-1} \tag{3.5.5}$$

It follows from $x_{\mu_1,\dots,\mu_N}\neq 0$ that $|\mu| = |\lambda|+\Delta$.

There exist a number $t\in\{k,k+1,\dots N-1\}$ such that $t\notin\{j_2,\dots,j_N\}$ for terms of RHS of (3.5.4). Then there is $t'\in\{k,k+1,\dots N-1\}$ such that $t'\notin\{-\mu_{k+1}+k,-\mu_{k+2}+k+1,\dots,-\mu_N+N-1\}$. Hence there is $\mu_s>0$ for $s\geq k+1$. Therefore $k+1\leq|\mu| = |\lambda|+\Delta$, which contradicts the assumption of the lemma. $\square$

Proof of Proposition 3.5.1. Using the notation from Propositions 3.4.4 and 3.4.5, we can rewrite (3.5.2b) as in formula (3.5.3). Lemma 3.5.1 implies that

$$\tilde P^{(N)}_{1,b}|\lambda\rangle = \sum_{k=-\lambda_1}^{|\lambda|-1-n'-nb} q^{(-k-nb-n')/n}\tilde\Psi_{k+n'+nb}\tilde\Phi^*_{-k}|\lambda\rangle \tag{3.5.6}$$

Propositions 3.4.4 and 3.4.5 imply that each term $q^{(-k-nb-n')/n}\tilde\Psi_{k+n'+nb}\tilde\Phi^*_{-k}$ stabilizes. $\square$

Convergence Action of operators $\tilde P^{(N)}_{1,b}$ does not weakly stabilize for $n'+nb>0$. Therefore we will need the following notion.

Definition 3.5.1. Action of operators $A^{(N)}\colon \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N}\to\mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N}$ *converges* if for any $w\in\mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N}$ the following sequence converges for $R\to\infty$

$$\varphi^{(m)}_{N+nR}\circ A^{(N+nR)}\circ\varphi^{(m)}_{N+nR,N}(w). \tag{3.5.7}$$

Remark 3.5.1. This is the first place in our article where we use that the base field is $\mathbb{C}$, but not a field of characteristic 0. Note, that $\Lambda^{\infty/2}_{v,m}\left(\mathbb{C}^n[Y^{\pm1}]\right)$ is graded vector space with finite dimensional graded components. Therefore the word 'converges' is understood in sence of finite dimensional vector space over $\mathbb{C}$.

Actually all convergence will be just convergence of infinite geometric series. Therefore all matrix elements at the end will be rational functions.

Lemma 3.5.2. $\hat\Psi_{k+\Delta+n}\hat\Phi^*_{-k-n}\varphi^{(0)}_N|\lambda\rangle = v^2\hat\Psi_{k+\Delta}\hat\Phi^*_{-k}\varphi^{(0)}_N|\lambda\rangle$ *for* $\Delta>0$ *and* $k\geq|\lambda|-\Delta$.

Proof. Let $\Delta = nl-s$ for $s=0,\dots,n-1$. Lemma 3.5.1 implies that for $k+nl\geq|\lambda|+(nl-\Delta)$ we have $\hat\Psi_{k+\Delta}\hat\Phi^*_{-k-nl}\varphi^{(0)}_N|\lambda\rangle=0$, and for $k+nl\geq|\lambda|-nl+(nl-\Delta)$ we have $\hat\Psi_{k+\Delta}\hat\Phi^*_{-k-nl}B_l\varphi^{(0)}_N|\lambda\rangle=0$. Hence for $k\geq|\lambda|-\Delta$ we get using the Proposition 3.4.8

$$0=[B_l,\hat\Psi_{k+\Delta}\hat\Phi^*_{-k-nl}]\varphi^{(0)}_N|\lambda\rangle = v^{2l(n-1)}\left(\hat\Psi_{k+\Delta+ln}\hat\Phi^*_{-k-ln}-v^{2l}\hat\Psi_{k+\Delta}\hat\Phi^*_{-k}\right)\varphi^{(0)}_N|\lambda\rangle. \tag{3.5.8}$$

We argue by induction on l. For $l=1$ the lemma follows from (3.5.8). Then for $j+n\geq|\lambda|-(\Delta-n)$

$$\begin{aligned}0&=[B_1,\hat\Psi_{j+\Delta+n}\hat\Phi^*_{-j-2n}]\varphi^{(0)}_N|\lambda\rangle - v^2[B_1,\hat\Psi_{j+\Delta}\hat\Phi^*_{-j-n}]\varphi^{(0)}_N|\lambda\rangle\\ &= v^{2(n-1)}\left(\hat\Psi_{j+\Delta+2n}\hat\Phi^*_{-j-2n}-2v^2\hat\Psi_{j+\Delta+n}\hat\Phi^*_{-j-n}+v^4\hat\Psi_{j+\Delta}\hat\Phi^*_{-j}\right)\varphi^{(0)}_N|\lambda\rangle.\end{aligned} \tag{3.5.9}$$

Relation (3.5.9) implies that for $k\geq|\lambda|-\Delta$

$$\left(\hat\Psi_{k+\Delta+ln}\hat\Phi^*_{-k-ln}-lv^{2l-2}\hat\Psi_{k+\Delta+n}\hat\Phi^*_{-k-n}+(l-1)v^{2l}\hat\Psi_{k+\Delta}\hat\Phi^*_{-k}\right)\varphi^{(0)}_N|\lambda\rangle=0. \tag{3.5.10}$$

The step of the induction follows from (3.5.8) and (3.5.10). $\square$

This lemma will be useful to prove that series (3.5.14) converges (essentially, this boils down to convergence of a geometric series). The next lemma is a finite analogue valid before the limit $N \to \infty$.

Lemma 3.5.3. $\tilde{\Psi}_{k+\Delta}\tilde{\Phi}^*_{-k}|\lambda\rangle = v^2\tilde{\Psi}_{k-n+\Delta}\tilde{\Phi}^*_{-k+n}|\lambda\rangle$ *for* $\Delta > 0$*,* $N - \Delta > k \geq |\lambda| - \Delta + n$*.*

Proof. Let $\tilde{\Phi}^*_{-k}|\lambda\rangle = \sum_\mu \tilde{x}_\mu e_{-\mu_1+1} \wedge e_{-\mu_2+2} \cdots \wedge e_{-\mu_{N-1}+N-1}$. We claim that

$$\Psi_{k+\Delta}\Big(e_{-\mu_1+1} \wedge e_{-\mu_2+2} \cdots \wedge e_{-\mu_{N-1}+N-1}\Big) = 0, \quad \text{if } l(\mu) \neq k+\Delta. \tag{3.5.11}$$

Indeed, if $\mu_{k+\Delta} = 0$, then we get zero from the vanishing of tale $e_{k+\Delta} \wedge \cdots \wedge e_{N-1} \wedge e_{k+\Delta}$ by Lemma 3.4.2.

If $\mu_{k+\Delta+1} > 0$ then there exist a number $t \in \{k+\Delta+1, k+\Delta+2, \ldots, N-1\}$ such that $t \notin \{-\mu_{k+\Delta+1}+k+\Delta+1, \ldots, -\mu_{N-1}+N-1\}$. Consider the expansion

$$\Psi_{k+\Delta}\Big(e_{-\mu_1+1} \wedge e_{-\mu_2+2} \cdots \wedge e_{-\mu_{N-1}+N-1}\Big) = \sum_{\nu_1 \geq \cdots \geq \nu_N} \bar{x}_{\nu_1,\ldots,\nu_N} e_{-\nu_1} \wedge e_{-\nu_2+1} \cdots \wedge e_{-\nu_N+N-1}. \tag{3.5.12}$$

Then for each term on the RHS there is $t' \in \{k+\Delta+1, k+\Delta+2, \ldots, N-1\}$ such that $t' \notin \{-\nu_{k+\Delta+2}+k+\Delta+1, \ldots, -\nu_N+N-1\}$. Hence there is $\nu_p > 0$ for $p > k+\Delta+1$. Therefore $k+\Delta+1 < |\nu| = |\lambda| - \Delta$, which contradicts the assumption of the lemma. Hence the expression (3.5.12) is zero.

Therefore

$$\begin{aligned}
\tilde{\Psi}_{k+\Delta}\tilde{\Phi}^*_{-k}|\lambda\rangle &= \tilde{\Psi}_{k+\Delta}\Big(\sum_{\mu,\, l(\mu)=k+\Delta} \tilde{x}_\mu e_{-\mu_1} \wedge \cdots \wedge e_{-\mu_{k+\Delta}+k+\Delta} \wedge e_{k+\Delta+1} \cdots \wedge \cdots \wedge e_{N-1}\Big) \\
&= (-1)^N v^{-\frac{n-1}{n}(N-1)} \sum_{\mu,\, l(\mu)=k+\Delta} \tilde{x}_\mu\, e_{-\mu_1} \wedge \cdots \wedge e_{-\mu_{k+\Delta}+k+\Delta} \wedge e_{k+\Delta+1} \wedge \cdots \wedge e_{N-1} \wedge e_{k+\Delta} \\
&= (-1)^{k+\Delta+1}\, v^{N-1-k-\Delta-\lfloor\frac{N-1-k-\Delta}{n}\rfloor-\frac{n-1}{n}(N-1)} \sum_{\mu,\, l(\mu)=k+\Delta} \tilde{x}_\mu\, e_{-\mu_1} \wedge \cdots \wedge e_{-\mu_{k+\Delta}+k+\Delta} \wedge e_{k+\Delta} \wedge e_{k+\Delta+1} \wedge \cdots \wedge e_{N-1}.
\end{aligned}$$

It was shown in Remark 3.4.1 that the coefficients $\tilde{x}_\mu$ are stable. Hence the product $\tilde{\Psi}_{k+\Delta}\tilde{\Phi}^*_{-k}$ is stable

$$\varphi^{(0)}_{N+n,N}\tilde{\Psi}_{k+\Delta}\tilde{\Phi}^*_{-k}|\lambda\rangle = \tilde{\Psi}_{k+\Delta}\tilde{\Phi}^*_{-k}\varphi^{(0)}_{N+n,N}|\lambda\rangle \tag{3.5.13}$$

Hence the lemma follows from Lemma 3.5.2. □

Proposition 3.5.2. *The operators* $\tilde{P}^{(N)}_{1,b} = v^{\frac{N}{n}} P^{(N)}_{1,b}$ *converge for* $|q^{-1}v^2| < 1$*. Moreover, the induced operator* $\hat{P}_{1,b}$ *equals*

$$\hat{P}_{1,b} = \sum_{k\in\mathbb{Z}} q^{(-k-nb-n')/n}\hat{\Psi}_{k+n'+nb}\hat{\Phi}^*_{-k}. \tag{3.5.14}$$

In particular, the series at the RHS of (3.5.14) *converges.*

Proof. It follows from (3.5.2b) that

$$\tilde{P}^{(N)}_{1,b} = \sum_{k\in\mathbb{Z}} q^{(-k-nb-n')/n}\tilde{\Psi}_{k+n'+nb}\tilde{\Phi}^*_{-k}. \tag{3.5.15}$$

We know that each term in right side of (3.5.15) stabilizes to the corresponding term in the right side of (3.5.14). Let us consider the vector

$$w_{k,R} = \varphi^{(0)}_{N+nR} \circ \Big(q^{(-k-nb-n')/n}\tilde{\Psi}_{k+n'+nb}\tilde{\Phi}^*_{-k}\Big)\,(e_{-\lambda_1} \wedge e_{-\lambda_r+r-1} \wedge e_r \wedge \cdots \wedge e_{N-1+nR}) \tag{3.5.16}$$

Stabilization mentioned above means that $w_{k,R}$ stablilizes for any fixed k and $R \to \infty$. Now the proposition follows from the following statements

(i) $w_{k,R} = 0$ for $k < -\lambda_1$

(ii) $w_{k,R} = q^{-1}v^2 w_{k-n,R}$ for $N + nR - n' - nb > k \geq |\lambda| - n' - nb + n$

(iii) $w_{k,R}$ =0 for $k \geq N + nR - n' - nb$

Statement (i) is obvious. Statement (ii) is equivalent to Lemma 3.5.3. For the statement (iii) note that terms containing $e_{k+n'+nb}$ vanishes after $\varphi^{(0)}_{N+nR}$ for such k). □

In the rest of this subsection we will assume $|q^{-1}v^2| < 1$. In order to study relations on $\hat{P}_{1,b_2}$ defined above we will need the following Proposition.

Proposition 3.5.3. *The operators $\tilde{P}^{(N)}_{1,b_t} \dots \tilde{P}^{(N)}_{1,b_2} \tilde{P}^{(N)}_{1,b_1}$ converge to $\hat{P}_{1,b_t} \dots \hat{P}_{1,b_2} \hat{P}_{1,b_1}$.*

Proof. Denote

$$|\mu,\nu\rangle = e_{-\mu_1} \wedge e_{-\mu_2+1} \wedge \dots \wedge e_{\mu_r+r-1} \wedge e_r \wedge \dots \wedge e_{N-s-1} \wedge e_{N-s+\nu_s} \dots \wedge e_{N-2+\nu_2} \wedge e_{N-1+\nu_1}. \quad (3.5.17)$$

In particular, $|\mu,\varnothing\rangle = |\mu\rangle$. Let

$$\tilde{P}^{(N)}_{1,b}|\lambda\rangle = \sum_{\mu} x^{(N)}_{\mu}|\mu\rangle + \sum_{\tilde{\mu},\tilde{\nu}} y^{(N)}_{\tilde{\mu},\tilde{\nu}}|\tilde{\mu},\tilde{\nu}\rangle \quad (3.5.18)$$

In the first sum we have $|\mu| = |\lambda| + n + n'b$. In the second sum we have $|\tilde{\mu}| - |\tilde{\nu}| = |\lambda| + n + n'b$ and $0 < |\tilde{\nu}| < n + n'b$. Note that only finitely many diagram μ, $\tilde{\mu}$ and $\tilde{\nu}$ satisfy this conditions.

Lemma 3.5.4. *The coefficients $y^{(N)}_{\tilde{\mu},\tilde{\nu}}$ tends to 0 for $N \to \infty$.*

Proof. Note that

$$\sum_{\tilde{\mu},\tilde{\nu}} y^{(N)}_{\tilde{\mu},\tilde{\nu}}|\tilde{\mu},\tilde{\nu}\rangle = \sum_{k=N-n'-nb}^{N-1} q^{(-k-nb-n')/n}\tilde{\Psi}_{k+n'+nb}\tilde{\Phi}^*_{-k}|\lambda\rangle \quad (3.5.19)$$

Also, we can see that

$$\tilde{\Phi}^*_{-k}|\lambda\rangle = \sum_{\Delta=0}^{N-1-k} \mathrm{y}^{(k,N)}_{\tilde{\mu},\Delta}\, e_{-\tilde{\mu}_1} \wedge e_{-\tilde{\mu}_2+1} \wedge \dots \wedge \hat{e}_{k+\Delta} \wedge \dots \wedge e_{N-1} \quad (3.5.20)$$

To study the coefficients $\mathrm{y}^{(k,N)}_{\tilde{\mu},\Delta}$ we will use the following trick. Let us act by $\Psi_{k+\Delta}$ on (3.5.20). Using Lemma 3.4.1, we obtain

$$\Psi_{k+\Delta}\tilde{\Phi}^*_{-k}|\lambda\rangle = (-v)^{N-1-k-\Delta}\, v^{-\lfloor \frac{N-1-k-\Delta}{n} \rfloor} \mathrm{y}^{(k,N)}_{\tilde{\mu},\Delta}\, |\tilde{\mu}\rangle \quad (3.5.21)$$

Recall (3.4.19). Denote $\mathsf{c}_N = (-1)^N v^{-\frac{n-1}{n}N}$. Then we have

$$\tilde{\Psi}_{k+\Delta}\tilde{\Phi}^*_{-k}|\lambda\rangle = (-v)^{N-1-k-\Delta}\, v^{-\lfloor \frac{N-1-k-\Delta}{n} \rfloor} \mathsf{c}_{N-1}\mathrm{y}^{(k,N)}_{\tilde{\mu},\Delta}\, |\tilde{\mu}\rangle \quad (3.5.22)$$

Denote $\tilde{\mathrm{y}}^{(k,N)}_{\tilde{\mu},\Delta} = (-v)^{N-1-k-\Delta}\, v^{-\lfloor \frac{N-1-k-\Delta}{n} \rfloor} \mathsf{c}_{N-1}\mathrm{y}^{(k,N)}_{\tilde{\mu},\Delta}$. Note that Lemma 3.5.3 and stabilization relation (3.5.13) imply

$$\tilde{\mathrm{y}}^{(k+n,N+n)}_{\tilde{\mu},\Delta} = v^2\tilde{\mathrm{y}}^{(k,N+n)}_{\tilde{\mu},\Delta} = v^2\tilde{\mathrm{y}}^{(k,N)}_{\tilde{\mu},\Delta}. \quad (3.5.23)$$

Then $\mathsf{c}_{N-1+n}\mathrm{y}^{(k+n,N+n)}_{\tilde{\mu},\Delta} = v^2\mathsf{c}_{N-1}\mathrm{y}^{(k,N)}_{\tilde{\mu},\Delta}$. Hence $y^{(N+n)}_{\tilde{\mu},\tilde{\nu}} = q^{-1}v^2 y^{(N)}_{\tilde{\mu},\tilde{\nu}}$. This finishes the proof since we have assumed $|q^{-1}v^2| < 1$. □

Let

$$\tilde{P}^{(N)}_{1,b}|\alpha,\beta\rangle = \sum_{\gamma} \tilde{x}^{(N)}_{\gamma}|\gamma\rangle + \sum_{\epsilon,\delta} \tilde{y}^{(N)}_{\epsilon,\delta}|\epsilon,\delta\rangle \tag{3.5.24}$$

Lemma 3.5.5. *The coefficients $\tilde{x}^{(N)}_{\gamma}$ and $\tilde{y}^{(N)}_{\epsilon,\delta}$ are bounded.*

Proof. In case of empty β, the lemma follows from Proposition 3.5.2 and Lemma 3.5.4. Below we will deduce the general case from the case of empty β.

Recall the operators b_j, see (3.4.5). Note that

$$|\alpha,\beta\rangle = \sum_{j_1,\dots j_l,\rho} z_{j_1,\dots j_l,\rho} b_{j_1}\dots b_{j_l}|\rho\rangle \tag{3.5.25}$$

Moreover, the coefficients $z_{j_1,\dots j_l,\rho}$ does not depend on N. Then lemma follows from the commutation relation $[b_j, \tilde{P}^{(N)}_{1,b}] = (q^j-1)\tilde{P}^{(N)}_{1,b+j}$. □

Let

$$\tilde{P}^{(N)}_{1,b_t}\dots\tilde{P}^{(N)}_{1,b_2}\tilde{P}^{(N)}_{1,b_1}|\lambda\rangle = \sum_{\zeta} \hat{x}^{(N)}_{\zeta}|\zeta\rangle + \sum_{\eta,\theta} \hat{y}^{(N)}_{\eta,\theta}|\eta,\theta\rangle \tag{3.5.26}$$

Lemma 3.5.6. *The coefficients $\hat{y}^{(N)}_{\eta,\theta}$ tends to 0.*

Proof. Follows from Lemmas 3.5.4 and 3.5.5 by induction on t. □

Let us proof Proposition 3.5.3 by induction on t. Let

$$\tilde{P}^{(N)}_{1,b_{t-1}}\dots\tilde{P}^{(N)}_{1,b_1}|\lambda\rangle = \sum_{\tilde{\zeta}} x^{(N)}_{\tilde{\zeta}}|\tilde{\zeta}\rangle + \sum_{\tilde{\eta},\tilde{\theta}} y^{(N)}_{\tilde{\eta},\tilde{\theta}}|\tilde{\eta},\tilde{\theta}\rangle \tag{3.5.27}$$

$$\hat{P}_{1,b_{t-1}}\dots\hat{P}_{1,b_1}\varphi^{(0)}_N|\lambda\rangle = \sum_{\tilde{\zeta}} x_{\tilde{\zeta}}\varphi^{(0)}_N|\tilde{\zeta}\rangle \tag{3.5.28}$$

Note, that $\varphi^{(0)}_N|\eta,\theta\rangle = 0$. Hence the assumption of the induction says that $x^{(N)}_{\tilde{\zeta}}$ tends to $x_{\tilde{\zeta}}$. Then Lemmas 3.5.5 and 3.5.6 imply that

$$\lim_{N\to\infty}\left(\varphi^{(0)}_N\tilde{P}^{(N)}_{1,b_t}\dots\tilde{P}^{(N)}_{1,b_1}|\lambda\rangle - \hat{P}_{1,b_t}\dots\hat{P}_{1,b_1}\varphi^{(0)}_N|\lambda\rangle\right) = \lim_{N\to\infty}\sum_{\tilde{\eta},\tilde{\theta}} y^{(N)}_{\tilde{\eta},\tilde{\theta}}\varphi^{(0)}_N\tilde{P}^{(N)}_{1,b_t}|\eta,\theta\rangle = 0 \tag{3.5.29}$$

□

Now let us drop the assumption $u^{-\frac{1}{n}}q^{\frac{n+1}{2n}} = 1$

Theorem 3.5.1. *For $|q^{-1}v^2| < 1$ the following formulas determine an action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ on the space $\Lambda^{\infty/2}_{v,m}\left(\mathbb{C}^n[Y^{\pm1}]\right)$*

$$c \mapsto v^{-n}, \tag{3.5.30a}$$

$$P_{0,-j} \mapsto q^j B_{-j}, \qquad P_{0,j} \mapsto \frac{q^j-1}{v^{-2j}q^j-1}v^{-jn}B_j, \tag{3.5.30b}$$

$$P_{1,b} \mapsto \hat{P}_{1,b} = u^{-\frac{1}{n}}q^{\frac{n+1}{2n}}\sum_{k\in\mathbb{Z}} q^{(-k-nb-n')/n}\hat{\Psi}_{k+n'+nb}\hat{\Phi}^*_{-k}, \tag{3.5.30c}$$

for $q_1 = q$, $q_2 = v^{-2}$.

Proof. In this proof we will denote by $P_{0,j}$ and $P_{1,b}$ the images, prescribed by (3.5.30b) and (3.5.30c). The proof is verification of relations (3.3.6a), (3.3.6b), (3.3.6d), and (3.3.6h).

Relation (3.3.6a) follows from (3.4.23)

$$[P_{0,j}, P_{0,-j}] = \frac{q^j - 1}{v^{-2j}q^j - 1} v^{-jn} q^j [B_j, B_{-j}] = j\frac{(1-q^j)(v^{-nj} - v^{nj})}{(1-q^{-j}v^{2j})(1-v^{-2j})}. \tag{3.5.31}$$

Equality in (3.5.30c) was proven in Proposition 3.5.2. For relation (3.3.6b) we will use formula with vertex operators and Proposition 3.4.8. For example, for $j > 0$

$$\begin{aligned}[P_{0,j}, P_{1,b}] &= u^{-\frac{1}{n}} q^{\frac{n+1}{2n}} \sum_{k\in\mathbb{Z}} q^{(-k-nb-n')/n} \Big([P_{0,j}, \hat{\Psi}_{k+n'+nb}] \hat{\Phi}^*_{-k} + \hat{\Psi}_{k+n'+nb} [P_{0,j}, \hat{\Phi}^*_{-k}] \Big) \\ &= \frac{q^j - 1}{v^{-2j}q^j - 1} v^{-jn} \left(q^j v^{2j(n-1)} - v^{2jn} \right) P_{1,b+j} = (q^j - 1) v^{jn} P_{1,b+j} \end{aligned} \tag{3.5.32}$$

Relations (3.3.6d) and (3.3.6h) follow from Proposition 3.5.3 and Corollary 3.3.4. Here we definition of $P_{1,b}$ as a $\hat{P}_{1,b}$ (i.e. interpretation as the limit of $P^{(N)}_{1,b}$).

□

Theorem 3.5.2. *The operators $\tilde{P}^{(N)}_{-1,b} = v^{-\frac{N}{n}} P^{(N)}_{-1,b}$ converge for $|qv^{-2}| < 1$. Let $\hat{P}_{-1,b}$ denote the induced operators. The following formulas determine an action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^-$ on $\Lambda^{\infty/2}_{v,m}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$*

$$c \mapsto v^{-n} \tag{3.5.33a}$$

$$P_{0,-j} \mapsto q^j B_{-j} \qquad\qquad P_{0,j} \mapsto \frac{q^j - 1}{v^{-2j}q^j - 1} v^{-jn} B_j \tag{3.5.33b}$$

$$P_{-1,b} \mapsto v^{-nb} \hat{P}_{-1,b} = u^{\frac{1}{n}} q^{\frac{n-1}{2n}} v^{-nb} \sum_{k\in\mathbb{Z}} q^{k/n} \hat{\Phi}_{k-n'+nb} \hat{\Psi}^*_{-k} \tag{3.5.33c}$$

for $q_1 = q$, $q_2 = v^{-2}$

Sketch of a proof. The results of this Subsection have a counterpart for $P_{-1,b}$, the proofs are analogous. Relations (3.3.6e), (3.3.6i) hold for both $\hat{P}_{-1,b}$ and $v^{-nb}\hat{P}_{-1,b}$. Proposition 3.4.8 implies relation (3.3.6c). For example, for $j > 0$

$$\begin{aligned}[P_{0,j}, P_{-1,b}] &= u^{\frac{1}{n}} q^{\frac{n-1}{2n}} v^{-nb} \frac{q^j - 1}{v^{-2j}q^j - 1} v^{-jn} \sum_{k\in\mathbb{Z}} q^{k/n} \Big([B_j, \hat{\Phi}_{k-n'+nb}] \hat{\Psi}^*_{-k} + \hat{\Phi}_{k-n'+nb} [B_j, \hat{\Psi}^*_{-k}] \Big) \\ &= \frac{q^j - 1}{v^{-2j}q^j - 1} \left(1 - q^j v^{-2j}\right) P_{-1,b+j} = (1 - q^j) P_{-1,b+j}. \end{aligned} \tag{3.5.34}$$

□

Currents For $\alpha = 0, 1, \ldots, n-1$ let us define the following currents (i.e. operator-valued formal power series)

$$\hat{\Phi}_{(\alpha)}(z) = \sum_{k\in\mathbb{Z}} \hat{\Phi}_{\alpha+nk} z^{-k}, \qquad \hat{\Psi}_{(\alpha)}(z) = \sum_{k\in\mathbb{Z}} \hat{\Psi}_{\alpha+nk} z^{-k}, \tag{3.5.35}$$

$$\hat{\Phi}^*_{(\alpha)}(z) = \sum_{k\in\mathbb{Z}} \hat{\Phi}^*_{-\alpha+nk} z^{-k}, \qquad \hat{\Psi}^*_{(\alpha)}(z) = \sum_{k\in\mathbb{Z}} \hat{\Psi}^*_{-\alpha+nk} z^{-k}. \tag{3.5.36}$$

Then (3.5.30c) and (3.5.33c) can be reformulated as follows

$$E(z) = u^{-\frac{1}{n}} q^{\frac{n+1}{2n}} \sum_{\alpha-\beta\equiv n' \bmod n} q^{-\frac{\alpha}{n}} z^{\frac{\beta-\alpha+n'}{n}} \hat{\Psi}_{(\alpha)}(qz) \hat{\Phi}^*_{(\beta)}(z) \tag{3.5.37}$$

$$F(z) = u^{\frac{1}{n}} q^{\frac{n-1}{2n}} \sum_{\alpha-\beta\equiv -n' \bmod n} q^{\frac{\beta}{n}} z^{\frac{\beta-\alpha-n'}{n}} \hat{\Phi}_{(\alpha)}(v^{jn}z) \hat{\Psi}^*_{(\beta)}(qv^{jn}z) \tag{3.5.38}$$

3.5.3 Analytic continuation

It follows from Lemma 3.5.2 that the series (3.5.30c) applied to any vector $|\lambda\rangle_\infty$ gives infinite geometric series, which can be written as a rational function. This is analytic continuation from the region $|q^{-1}v^2| < 1$ to arbitrary q, v.

Another way to say this is just rewrite

$$\sum_{k\in\mathbb{Z}} q^{\frac{-k-nb-n'}{n}} \hat{\Psi}_{k+n'+nb}\hat{\Phi}^*_{-k} = \frac{1}{1-qv^{-2}} \sum_{k\in\mathbb{Z}} q^{\frac{-k-nb-n'}{n}} \left(\hat{\Psi}_{k+n'+nb}\hat{\Phi}^*_{-k} - v^{-2}\hat{\Psi}_{k+n'+nb+n}\hat{\Phi}^*_{-k-n}\right). \tag{3.5.39}$$

Lemma 3.5.2 implies that for any vector $|\lambda\rangle_\infty$, only finitely many terms of RHS of (3.5.39) does not annihilate $|\lambda\rangle$. Hence the sum is well-defined without the assumption $|qv^{-2}| < 1$.

Proposition 3.5.4. *The following formulas determine an action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ on $\Lambda^{\infty/2}_{v,m}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$*

$$c \mapsto v^{-n} \tag{3.5.40a}$$

$$P_{0,-j} \mapsto q^j B_{-j} \qquad P_{0,j} \mapsto \frac{q^j-1}{v^{-2j}q^j-1} v^{-jn} B_j \tag{3.5.40b}$$

$$P_{1,b}|\lambda\rangle = \frac{u^{-\frac{1}{n}}q^{\frac{n+1}{2n}}}{1-qv^{-2}} \sum_{k\in\mathbb{Z}} q^{\frac{-k-nb-n'}{n}} \left(\hat{\Psi}_{k+n'+nb}\hat{\Phi}^*_{-k} - v^{-2}\hat{\Psi}_{k+n'+nb+n}\hat{\Phi}^*_{-k-n}\right)|\lambda\rangle \tag{3.5.40c}$$

for $q_1 = q$, $q_2 = v^{-2}$.

Proof. Note that (3.5.40c) is an analytic continuation of (3.5.30c). The relations hold after the analytic continuation. □

The above construction can be reformulated in the language of currents. Namely, the following current is well-defined

$$\hat{\Psi}_{(\alpha)}(qz)\hat{\Phi}^*_{(\beta)}(z) = \frac{1}{1-qv^{-2}}(1-v^{-2}w/z)\hat{\Psi}_{(\alpha)}(w)\hat{\Phi}^*_{(\beta)}(z)\Big|_{w=qz} \tag{3.5.41}$$

Using this notation, we can use (3.5.37) without the assumption $|qv^{-2}| < 1$. Analogously, the following current is well defined

$$\hat{\Phi}_{(\alpha)}(v^nz)\hat{\Psi}^*_{(\beta)}(qv^nz) = \frac{1}{1-q^{-1}v^2}(1-v^2z/w)\hat{\Phi}_{(\alpha)}(v^nz)\hat{\Psi}^*_{(\beta)}(v^nw)\Big|_{w=qz} \tag{3.5.42}$$

We prefer to use the current form of the formulas. The next proposition is a counterpart of Proposition 3.5.4 for $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^-$. We omit not-current form version.

Proposition 3.5.5. *The following formulas determine an action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^-$ on $\Lambda^{\infty/2}_{v,m}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$*

$$c \mapsto v^{-n} \tag{3.5.43a}$$

$$P_{0,-j} \mapsto q^j B_{-j} \qquad P_{0,j} \mapsto \frac{q^j-1}{v^{-2j}q^j-1} v^{-jn} B_j \tag{3.5.43b}$$

$$F(z) \mapsto u^{\frac{1}{n}}q^{\frac{n-1}{2n}} \sum_{\alpha-\beta\equiv -n' \bmod n} q^{\frac{\beta}{n}} z^{\frac{\beta-\alpha-n'}{n}} \hat{\Phi}_{(\alpha)}(v^nz)\hat{\Psi}^*_{(\beta)}(qv^nz) \tag{3.5.43c}$$

for $q_1 = q$, $q_2 = v^{-2}$.

3.5.4 Example $n = 1$

In this case $[B_k, B_l] = k\delta_{k+l,0}$. The space $\Lambda^{\infty/2}_{v,m}\left(\mathbb{C}[Y^{\pm 1}]\right)$ is Fock space for the Heisenberg algebra. Namely it has a cyclic vector $|m\rangle = e_{-m} \wedge e_{-m+1} \wedge \dots$ such that $B_k|m\rangle = 0$ for $k > 0$. Let us consider an operator $e^{\pm Q}\colon \Lambda^{\infty/2}_{v,m}\left(\mathbb{C}[Y^{\pm 1}]\right) \to \Lambda^{\infty/2}_{v,m\pm 1}\left(\mathbb{C}[Y^{\pm 1}]\right)$ determined by $e^{\pm Q}|m\rangle = |m \pm 1\rangle$ and $[B_k, e^{\pm Q}] = 0$.

Proposition 3.5.6. *The operators* $\hat{\Phi}(z), \hat{\Psi}(z)\colon \Lambda^{\infty/2}_{v,m}\left(\mathbb{C}[Y^{\pm 1}]\right) \to \Lambda^{\infty/2}_{v,m+1}\left(\mathbb{C}[Y^{\pm 1}]\right)$ *are given by*

$$\hat{\Phi}(z) = \hat{\Psi}(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} B_{-k}\right) \exp\left(-\sum_{k=1}^{\infty} \frac{z^{-k}}{k} B_k\right) e^Q z^{m+1} \tag{3.5.44}$$

The operators $\hat{\Phi}^*(z), \hat{\Psi}^*(z)\colon \Lambda^{\infty/2}_{v,m}\left(\mathbb{C}[Y^{\pm 1}]\right) \to \Lambda^{\infty/2}_{v,m-1}\left(\mathbb{C}[Y^{\pm 1}]\right)$ *are given by*

$$\hat{\Phi}^*(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{v^{2k} z^k}{k} B_{-k}\right) \exp\left(\sum_{k=1}^{\infty} \frac{z^{-k}}{k} B_k\right) e^{-Q} z^{-m} \tag{3.5.45}$$

$$\hat{\Psi}^*(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{v^{-2k} z^k}{k} B_{-k}\right) \exp\left(\sum_{k=1}^{\infty} \frac{z^{-k}}{k} B_k\right) e^{-Q} z^{-m} \tag{3.5.46}$$

Proof. Follows from Proposition 3.4.8. □

Then

$$\hat{\Psi}(w)\hat{\Phi}^*(z) = \frac{w^m z^{-m}}{1 - v^2 z/w} \exp\left(\sum_{k=1}^{\infty} \frac{w^k - v^{2k} z^k}{k} B_{-k}\right) \exp\left(\sum_{k=1}^{\infty} \frac{z^{-k} - w^{-k}}{k} B_k\right) \tag{3.5.47}$$

$$\hat{\Phi}(z)\hat{\Psi}^*(w) = \frac{z^m w^{-m}}{1 - v^{-2} z/w} \exp\left(\sum_{k=1}^{\infty} \frac{z^k - v^{-2k} w^k}{k} B_{-k}\right) \exp\left(\sum_{k=1}^{\infty} \frac{w^{-k} - z^{-k}}{k} B_k\right) \tag{3.5.48}$$

Substituting this to (3.5.37) and (3.5.38), we obtain

$$E(z) = \frac{v u^{-1} q^{m+1}}{1 - q^{-1} v^2} \exp\left(\sum_{k=1}^{\infty} \frac{q^k - v^{2k}}{k} B_{-k} z^k\right) \exp\left(\sum_{k=1}^{\infty} \frac{1 - q^{-k}}{k} B_k z^{-k}\right) \tag{3.5.49}$$

$$F(z) = \frac{v^{-1} u q^{-m}}{1 - q v^{-2}} \exp\left(\sum_{k=1}^{\infty} \frac{v^k - q^k v^{-k}}{k} B_{-k} z^k\right) \exp\left(\sum_{k=1}^{\infty} \frac{v^{-k}(q^{-k} - 1)}{k} B_k z^{-k}\right) \tag{3.5.50}$$

The following proposition is [FHH+09, Prop. A.6].

Proposition 3.5.7. *Formulas* (3.5.49), (3.5.50) *and*

$$c \mapsto v^{-1} \qquad P_{0,-j} \mapsto q^j B_{-j} \qquad P_{0,j} \mapsto \frac{q^j - 1}{v^{-2j} q^j - 1} v^{-jn} B_j \tag{3.5.51}$$

determine an action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ *for* $q_1 = q$, $q_2 = v^{-2}$.

For $m = 0$ we will denote the representation by $\mathcal{F}_u$

Remark 3.5.2. Note that Propositions 3.5.4 and 3.5.5 guarantee only existence of actions of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ and $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^-$ separately. Remarkably, the actions of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^\pm$ are restrictions of the action of whole $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$. Below we will prove the same result for general n.

The obtained representation is celebrated *Fock representation* of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$. It was constructed in [FHH+09] via formulas (3.5.49), (3.5.50) (up to different notation). Though, for the best of our knowledge, the interpretation via the operators $\hat{\Phi}(z)$, $\hat{\Psi}(z)$, $\hat{\Phi}^*(z)$, and $\hat{\Psi}^*(z)$ is new.

3.6 Semi-infinite construction of twisted Fock module II

In the previous section, we have obtained actions of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ and $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^-$ on $\Lambda^{\infty/2}_{v,0}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$. In this section we prove that these actions (after a simple rescaling) give action of the whole algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$. We do not check directly the relations (3.3.6f), (3.3.6g) due to technical difficulties.

We prove that the defined below operators $\tilde{P}^{(N)}_{a,b}$ stabilize for $an'+bn \leq 0$ and general q and v. Therefore we obtain a representation of the corresponding subalgebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\swarrow} \subset U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ (subsection 3.6.1). We extend the action to the whole $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$, the obtained representation is isomorphic to twisted Fock module $\mathcal{F}^{\sigma}_u$ by construction. Then we compare the obtained action $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1) \curvearrowright \Lambda^{\infty/2}_{v,0}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$ with the actions of the subalgebras $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\pm}$.

Recall that the actions of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\pm}$ are determined by explicit formulas for Chevalley generators (Proposition 3.5.4 and 3.5.5). Hence we get explicit formulas for the action of Chevalley generators of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ on $\Lambda^{\infty/2}_{v,0}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$, the obtained representation is isomorphic to twisted Fock module $\mathcal{F}^{\sigma}_u$ (Theorem 3.6.1). This is the central result of the whole paper.

3.6.1 The limit for the bottom half

Existence of the limit Below we will use the results of Sections 3.2.2 and 3.2.3. Recall that m, m' are integers such that $nm'-n'm=1$ and $0 \leq m < n$, $0 \leq m' < n'$. Recalll also definition of $\sigma \in \widetilde{SL}(2,\mathbb{Z})$ in Section 3.2.2.

Lemma 3.6.1. *a) The operators $v^{\frac{kmN}{n}} P^{(N)}_{km,-km'}$ stabilize for $k \in \mathbb{Z}_{>0}$.*

b) The operators $v^{kN} P^{(N)}_{kn,-kn'} - u^{-k} v^{-k} q^{2k} \sum_{i=1}^{N} (v^2 q^{-1})^{ik}$ stabilize for $k \in \mathbb{Z}_{>0}$.

c) The operators $v^{-kN} P_{-kn,kn'} - u^{k} v^{k} q^{-k} \sum_{i=1}^{N} (v^{-2} q)^{ik}$, for $k \in \mathbb{Z}_{>0}$ stabilize.

Proof. a) The formula (3.3.12a) and $SL(2,\mathbb{Z})$-transformation properties of the $P^{(N)}_{a,b}$ generators imply

$$P^{(N)}_{km,-km'} = \sigma\left(P^{(N)}_{0,-k}\right) = q^k \mathbf{S}_- \sum_{i=1}^{N} \sigma\left(Y_i\right)^{-k} \mathbf{S}_- = q^k \mathbf{S}_- \sum_{i=1}^{N} \mathbf{A}_i^{-k} \mathbf{S}_-. \tag{3.6.1}$$

Note that $\sum_{i=1}^{N} \mathbf{A}_i^{-k}$ commutes with finite Hecke algebra and, in particular, with $\mathbf{S}_-$. Hence

$$P^{(N)}_{km,-km'} |\lambda\rangle = q^k \sum_{i=1}^{N} \mathbf{S}_- \mathbf{A}_i^{-k} \big(e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1}\big) \tag{3.6.2}$$

We decompose the proof into two steps.

Step 1. First we show that i-th term in (3.6.2) vanishes for $i > |\lambda| + k$. Denote

$$\mathbf{A}_i^{-k} \big(e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1}\big) = \sum_{j_1,\dots,j_N} c_{j_1,\dots j_N} e_{j_1} \otimes \cdots \otimes e_{j_N} \tag{3.6.3}$$

For a sequence $\{j_1,\dots,j_N\}$, we will say that a number r is a *hole* if $0 \leq r \leq N-1$ and $r \notin \{j_1,\dots,j_N\}$. Let us prove that for each summand in (3.6.3) there is a hole $r \geq i-1$.

We will use the formula (3.2.24). Note that the operators $\kappa_i^{-c} G^{\pm 1}_{i,j} \kappa_i^c$, $\kappa_i^{-c} G^{\pm 1}_{j,i} \kappa_i^c$, and κ_i preserve the existence of a hole with position $\geq i-1$. If there is no such holes, operator κ_i must create one. Hence the operator $\mathbf{A}_i^{-k}$ must create a hole.

Let us also denote

$$\mathbf{S}_- \mathbf{A}_i^{-k} \big(e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1}\big) = \sum_{\mu_1 \geq \mu_2 \geq \dots} \tilde{c}_\mu e_{-\mu_1} \wedge e_{-\mu_2+1} \wedge \cdots \wedge e_{N-1} \tag{3.6.4}$$

Note that $|\mu| = |\lambda| + k$. Existence of a hole r imply $\mu_{r+1} > 0$. Hence $|\mu| \geq r + 1 \geq i > |\lambda| + k$. Hence the sum runs over the empty set, i.e. $\mathbf{S}_-\mathbf{A}_i^{-k}\big(e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1}\big) = 0$.

Step 2. It remains to study the terms in (3.6.2) for small i. If we replace $N \mapsto N + n$ and use again formula (3.2.24), we get mk additional factors of the form $\kappa_i^{-c} G_{i,N+1} \dots G_{i,N+n} \kappa_i^c$. Each of this factors acts diagonally with addition of the terms with holes $r \geq N$. As before, such additional terms vanish after the action of $\mathbf{S}_-$. The diagonal part acts by v^{-mk} by the formulas (3.2.16)–(3.2.18).

b) The proof is similar to the previous one. Using formula (3.3.12b) and $SL(2,\mathbb{Z})$-transformation property, we obtain $P^{(N)}_{kn,-kn'} = q^k \mathbf{S}_- \sum_{i=1}^N \mathbf{B}_i^{-k} \mathbf{S}_-$. Then

$$v^{kN} P^{(N)}_{kn,-kn'} |\lambda\rangle = q^k v^{kN} \sum_{i=1}^{N} \mathbf{S}_- \mathbf{B}_i^{-k} \big(e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1}\big) \tag{3.6.5}$$

We have two steps as in the proof above.

Step 1. Let $i > |\lambda|$. Hence $\lambda_i = 0$. In order to compute i-th term in the sum (3.6.5) we use the formula (3.2.20). Each triangular operator of the form $\kappa_i^{-c} G^{\pm 1}_{i,j} \kappa_i^c$, $\kappa_i^{-c} G^{\pm 1}_{j,i} \kappa_i^c$ acts on $e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1}$ diagonally with addition of the terms with holes $r \geq i - 1$. Terms with holes vanishes after the action of $\mathbf{S}_-$. The diagonal contribution was computed in the proof of the Corollary 3.2.3. Hence the i-th term in the sum (3.6.5) is equal to

$$\begin{aligned} q^k v^{kN} (u_0 \dots u_{n-1} q^{1-n} v^{\mathtt{1}_i^{(\lambda)}} q^{-\lambda_i + i - 1})^{-k} \big(e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1}\big) \\ = u^{-k} v^{-k} q^{2k} (v^2 q^{-1})^{ik} \big(e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1}\big) \end{aligned}$$

where we used $\mathtt{1}_i^{(\lambda)} = N + 1 - 2i$ and convention (3.2.29).

Step 2. The same as above.

c) Using the formula (3.3.12b) and $SL(2,\mathbb{Z})$ invariance of the $P_{a,b}$ generators we have $P^{(N)}_{-kn,kn'} = \mathbf{S}_- \sum_{i=1}^N \mathbf{B}_i^k \mathbf{S}_-$. The remaining proof is similar to the proof of b). $\square$

Proposition 3.6.1. *The operators $v^{\frac{aN}{n}} P^{(N)}_{a,b}$ stabilize for $an' + bn < 0$.*

Proof. It follows from the commutation relations that any $P_{a,b} \in U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ with $a > 0$ can be represented as algebraic combination of $P_{1,0}$, its commutators with $P_{0,b}$ for $b \in \mathbb{Z}$ and also c, c'. Using $\widetilde{SL}(2,\mathbb{Z})$ symmetry we see that any element $P_{a,b} \in U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ with $an' + bn < 0$ is algebraic combination of $P_{m,-m'}$, its commutators with $P_{bn,-bn'}$, $b \in \mathbb{Z}$ and c, c'. To finish the proof we use Lemma 3.6.1. $\square$

Remark 3.6.1. a) This proposition gives another proof of the Proposition 3.5.1.

b) The additional series $\sum_{i=1}^{\infty} (v^2 q^{-1})^{ki}$ which appears in Lemma 3.6.1b) converges if $|v^2 q^{-1}| < 1$. This is in agreement with Theorem 3.5.1.

For $an' + bn < 0$, let $\tilde{P}^{(N)}_{a,b} = v^{\frac{aN}{n}} P^{(N)}_{a,b}$. Denote stable limit of $\tilde{P}^{(N)}_{a,b}$ by $\hat{P}_{a,b}$. Similarly consider operators

$$\tilde{P}^{(N)}_{kn,-kn'} = v^{kN} P^{(N)}_{kn,-kn'} - u^{-k} v^{-k} q^{2k} \sum_{i=1}^{N} (v^2 q^{-1})^{ik} + \frac{u^{-k} v^k q^k}{1 - (v^2 q^{-1})^k} \tag{3.6.6}$$

$$\tilde{P}^{(N)}_{-kn,kn'} = v^{-kN} P^{(N)}_{-kn,kn'} - u^k v^k q^{-k} \sum_{i=1}^{N} (v^{-2} q)^{ik} + \frac{u^k v^{-k}}{1 - (v^{-2} q)^k} \tag{3.6.7}$$

Denote by $\hat{P}_{kn,-kn'}$ and $\hat{P}_{-kn,kn'}$ stable limit of $\tilde{P}^{(N)}_{kn,-kn'}$ and $\tilde{P}^{(N)}_{-kn,kn'}$. Let $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\swarrow}$ be a subalgebra of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ generated by $P_{a,b}$ for $an' + bn \leq 0$.

Proposition 3.6.2. *There is an action of* $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\swarrow}$ *given by*

$$c \mapsto v^{-n}, \qquad c' \mapsto v^{-n'}, \tag{3.6.8a}$$

$$P_{a,b} \mapsto \hat{P}_{a,b} \qquad \textit{for } a \geq 0, \tag{3.6.8b}$$

$$P_{a,b} \mapsto v^{-nb-n'a}\hat{P}_{a,b} \qquad \textit{for } a < 0. \tag{3.6.8c}$$

Proof. From the limit arguments we see that operators $\hat{P}_{a,b}$ satisfy relations of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\swarrow}$ for $c = c' = 1$. It remains to show that formulas (3.6.8b)–(3.6.8c) defines isomorphism of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\swarrow}\big|_{c=1,c'=1}$ and $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\swarrow}\big|_{c=v^{-n},c'=v^{-n'}}$.

Let $\tilde{\sigma}$ be an element of $\widetilde{SL}(2,\mathbb{Z})$ such that corresponding matrix in $SL(2,\mathbb{Z})$ is $\begin{pmatrix} -n' & -n \\ m' & m \end{pmatrix}$ and $n_{\tilde{\sigma}}(0,-1) = 0$. Using the formula (3.3.2) we see action of $\tilde{\sigma}$ induces an isomorphism

$$U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\swarrow}\big|_{c=v^{-n},c'=v^{-n'}} \xrightarrow{\sim} U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{+}\big|_{c=1,c'=v}. \tag{3.6.9}$$

In the region $an' + bn \leq 0$ the winding number $n_{\tilde{\sigma}}(a,b) = 0$ for $a \geq 0$ and $n_{\tilde{\sigma}}(a,b) = 1$ for $a < 0$, hence the formula (3.3.3) for the action of $\tilde{\sigma}$ on $P_{a,b}$ generators gives

$$P_{a,b} \mapsto \begin{cases} P_{-n'a-nb,m'a+mb} & \text{for } a \geq 0 \\ v^{-n'a-nb}P_{-n'a-nb,m'a+mb} & \text{for } a < 0 \end{cases} \tag{3.6.10}$$

Similarly $\tilde{\sigma}$ induces an isomorphism between $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\swarrow}\big|_{c=1,c'=1}$ and $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{+}\big|_{c=1,c'=1}$

$$P_{a,b} \mapsto P_{-n'a-nb,m'a+mb}. \tag{3.6.11}$$

Since the relations of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{+}$ does not include c', there is an isomorphism of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{+}\big|_{c=1,c'=v}$ and $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{+}\big|_{c=1,c'=1}$ which acts $P_{a,b} \mapsto P_{a,b}$ for any $a \geq 0$ and b.

To sum up the above, we have obtained a chain of isomorphisms

$$U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\swarrow}\big|_{c=v^{-n},c'=v^{-n'}} \xrightarrow{\sim} U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{+}\big|_{c=1,c'=v} \xrightarrow{\sim} U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{+}\big|_{c=1,c'=1} \xrightarrow{\sim} U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\swarrow}\big|_{c=1,c'=1}.$$

To finish the proof we notice that the composition indeed is given by (3.6.8b)–(3.6.8c). ☐

Denote the obtained representation by $\mathcal{F}_u^{\swarrow}$.

Connection with twisted representation Below we will give an alternative interpretation of $\mathcal{F}_u^{\swarrow}$ as a version of twisted representation. To do this we need to introduce the following notions.

Let $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\downarrow}$ be a subalgebra of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ generated by $P_{a,b}$ for $b \leq 0$. In Subsection 3.5.4 we have constructed action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ on $\Lambda^{\infty/2}_{v,0}\left(\mathbb{C}[Y^{\pm 1}]\right)$. Denote its restriction to $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\downarrow}$ by ρ_1. On the other hand, let us consider particular case $n = 1$ and $n' = 0$ of Proposition 3.6.2. It gives *a priori* another action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\downarrow}$ on $\Lambda^{\infty/2}_{v,0}\left(\mathbb{C}[Y^{\pm 1}]\right)$. Denote it by ρ_2.

Lemma 3.6.2. *The actions* ρ_1 *and* ρ_2 *coincide.*

Proof. Let us prove $\rho_1(P_{a,b}) = \rho_2(P_{a,b})$ for $a \geq 0$. The operator $v^{\frac{aN}{n}}P^{(N)}_{a,b}$ converges to $\rho_1(P_{a,b})$ for $|v^2q^{-1}| < 1$ by Proposition 3.5.3. On the other hand, operator $\tilde{P}^{(N)}_{a,b}$ stabilizes and the induced operator is $\rho_2(P_{a,b})$. Notice that for $|v^2q^{-1}| < 1$, the limits of $v^{\frac{aN}{n}}P^{(N)}_{a,b}$ and $\tilde{P}^{(N)}_{a,b}$ coincide (even for $an' + bn = 0$, see (3.6.6)–(3.6.7)). Hence $\rho_1(P_{a,b}) = \rho_2(P_{a,b})$ for $|v^2q^{-1}| < 1$. Since matrix coefficients of $\rho_1(P_{a,b})$ and $\rho_2(P_{a,b})$ are analytic (even rational) functions of q and v, we have $\rho_1(P_{a,b}) = \rho_2(P_{a,b})$ for any values of q and v.

The case $a < 0$ is analogous. Note that prefactors in formulas (3.6.8c) and (3.5.33c) agrees for $n = 1, n' = 0$. ☐

Let $\mathcal{F}_u^{\downarrow}$ be the restriction of Fock module $\mathcal{F}_u$ to $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\downarrow}$. Recall that σ is an element in $\widetilde{SL}(2,\mathbb{Z})$ such that $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\swarrow} = \sigma\left(U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\downarrow}\right)$. Recall the formulas (3.3.2) and (3.3.3) for the action $\widetilde{SL}(2,\mathbb{Z}) \curvearrowright U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$.

Lemma 3.6.3. *For $b \leq 0$, the element $\sigma(P_{a,b})$ acts on $\mathcal{F}_u^{\swarrow}$ as*

$$v^{-n_\sigma(1,0)b} P_{na-mb,-n'a+m'b} \qquad \textit{for } a>0 \textit{ or } \frac{m}{n}b \geq a \tag{3.6.12}$$

$$v^{-(n_\sigma(1,0)+1)b} P_{na-mb,-n'a+m'b} \qquad \textit{otherwise} \tag{3.6.13}$$

Proof. In this proof we assume $b \leq 0$. One can check that the winding number is given by

$$n_\sigma(a,b) = \begin{cases} n_\sigma(1,0) & \text{for } a>0 \text{ or } \frac{m}{n}b \geq a \\ n_\sigma(1,0)+1 & \text{otherwise} \end{cases} \tag{3.6.14}$$

To finish the proof, we notice that the element $(c')^{na-mb}\, c^{-n'a+m'b}$ acts on $\mathcal{F}_u^{\swarrow}$ as v^{-b}. $\square$

Proposition 3.6.3. *There is an isomorphism of vector spaces $\hat{\psi}\colon \mathcal{F}_u^{\downarrow} \to \mathcal{F}_u^{\swarrow}$ such that $\hat{\psi}$ intertwine the actions. More precisely, $\sigma(X)\hat{\psi}w = \hat{\psi}\,(Xw)$ for any $w \in \mathcal{F}_u^{\downarrow}$ and $X \in U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\downarrow}$.*

Proof. Note that $\mathcal{F}_u^{\downarrow}$ is a free cyclic module over the algebra generated by $P_{0,-k}$ for $k \in \mathbb{Z}_{>0}$ with cyclic vector $|0\rangle = e_0 \wedge e_1 \wedge \dots$. Analogously, $\mathcal{F}_u^{\swarrow}$ is a free cyclic over algebra generated by $\sigma(P_{0,-k})$ with the corresponding[3] cyclic vector $|0\rangle$. Hence there is an isomorphism of vector spaces $\hat{\psi}\colon \mathcal{F}_u^{\downarrow} \to \mathcal{F}_u^{\swarrow}$ such that $\hat{\psi}(|0\rangle) = |0\rangle$ and $\hat{\psi}$ satisfy the intertwining property for $X = P_{0,-k}$.

It remains to prove that the intertwining property holds for any $X \in U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^{\downarrow}$. To do this we will need the following notation. Let $\mathcal{F}_u^{\downarrow}[\leq k]$ and $\mathcal{F}_u^{\swarrow}[\leq k]$ be subspaces, spanned by $|\lambda\rangle_\infty$ for $|\lambda| \leq k$. Analogously, let $\mathbf{S}_-\left(\mathbb{C}[Y^{\pm 1}]\right)^{\otimes N}[\leq k]$ and $\mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}[\leq k]$ be subspaces, spanned by $|\lambda\rangle_N = e_{-\lambda_1} \wedge e_{-\lambda_2+1} \wedge \dots \wedge e_{N-1}$ for $|\lambda| \leq k$. Let us consider the following diagram

$$\begin{tikzcd} \mathbf{S}_-\left(\mathbb{C}[Y^{\pm 1}]\right)^{\otimes N}[\leq k] \arrow[r, "\varphi_N^{[1]}"] \arrow[d, "\tilde{\psi}"] & \mathcal{F}_u^{\downarrow}[\leq k] \arrow[d, "\hat{\psi}"] \\ \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}[\leq k] \arrow[r, "\varphi_N^{[n]}"] & \mathcal{F}_u^{\swarrow}[\leq k] \end{tikzcd} \tag{3.6.15}$$

Recall that we used notation $\varphi_N^{(m)}$ for the canonical map from the definition inductive limit (see subsection 3.4.2). In this proof we omit superscript (m) since we consider only the case $m=0$. We denote the corresponding maps by $\varphi_N^{[1]}$ and $\varphi_N^{[n]}$ to distinguish the first and the second rows of (3.6.15). Note that here we have used interpretation of $\mathcal{F}_u^{\downarrow}$ via ρ_2, see Lemma 3.6.2.

The map $\tilde{\psi}$ induced from the isomorphism $\psi\colon \mathbf{C}_u^{\sigma} \to \mathbf{C}_u^{(n,n')}$, see Theorem 3.2.4. More precisely, it is determined by

$$\tilde{\psi}|0\rangle_N = |0\rangle_N \qquad v^{-n_\sigma(1,0)b}\sigma\left(P_{a,b}^{(N)}\right)\tilde{\psi} = \tilde{\psi}P_{a,b}^{(N)} \tag{3.6.16}$$

Also, note that we abuse notation using the same symbols for maps with and without restriction to corresponding $[\leq k]$ subspace.

Lemma 3.6.4. *Diagram* (3.6.15) *is commutative for any (fixed) k and sufficiently large N.*

[3]Here and below we use the same notation for vectors in $\mathcal{F}_u^{\downarrow}$ and $\mathcal{F}_u^{\swarrow}$. Hopefully, this will not lead to a confusion.

Proof. One can verify that the operators $\varphi_N^{[n]}\tilde{\psi}$ and $\hat{\psi}\varphi_N^{[1]}$ for sufficiently large N

$$\varphi_N^{[n]}\tilde{\psi}|0\rangle_N = |0\rangle \qquad \sigma(P_{0,-l})\,\varphi_N^{[n]}\tilde{\psi}|\lambda\rangle_N = \varphi_N^{[n]}\tilde{\psi}\tilde{P}_{0,-l}^{(N)}|\lambda\rangle_N \tag{3.6.17}$$

$$\hat{\psi}\varphi_N^{[1]}|0\rangle_N = |0\rangle \qquad \sigma(P_{0,-l})\,\hat{\psi}\varphi_N^{[1]}|\lambda\rangle_N = \hat{\psi}\varphi_N^{[1]}\tilde{P}_{0,-l}^{(N)}|\lambda\rangle_N \tag{3.6.18}$$

Actually (3.6.17) and (3.6.18) are the same property for the operators $\varphi_N^{[n]}\tilde{\psi}$ and $\hat{\psi}\varphi_N^{[1]}$ correspondingly. To finish the proof we note that the maps $\varphi_N^{[n]}\tilde{\psi}$ and $\hat{\psi}\varphi_N^{[1]}$ are determined by the property. □

Let us prove that $\sigma(P_{a,b})\hat{\psi}|\lambda\rangle = \hat{\psi}\,(P_{a,b}|\lambda\rangle)$ for $b \le 0$. Let us take k large enough such that $P_{a,b}|\lambda\rangle \in \mathcal{F}_u^{\downarrow}[\le k]$. Then we take sufficiently large N such that diagram (3.6.15) is commutative and

$$P_{a,b}\varphi_N^{[1]}|\lambda\rangle_N = \begin{cases} \varphi_N^{[1]}\tilde{P}_{a,b}^{(N)}|\lambda\rangle_N & \text{for } a \ge 0 \\ v^{-b}\varphi_N^{[1]}\tilde{P}_{a,b}^{(N)}|\lambda\rangle_N & \text{otherwise} \end{cases} \tag{3.6.19}$$

Note that here we used the interpretation of the action as ρ_2, see Lemma 3.6.2. Also we take N large enough such that

$$P_{a,b}\varphi_N^{[n]}|\lambda\rangle_N = \begin{cases} \varphi_N^{[n]}\tilde{P}_{a,b}^{(N)}|\lambda\rangle_N & \text{for } a \ge 0 \\ v^{-n'a-nb}\varphi_N^{[n]}\tilde{P}_{a,b}^{(N)}|\lambda\rangle_N & \text{otherwise} \end{cases} \tag{3.6.20}$$

Using Lemma 3.6.3, we obtain

$$\sigma(P_{a,b})\,\varphi_N^{[n]}|\lambda\rangle_N = \begin{cases} v^{-n_\sigma(1,0)b}\varphi_N^{[n]}\tilde{P}_{na-mb,-n'a+m'b}^{(N)}|\lambda\rangle_N & \text{for } a \ge 0 \\ v^{-(n_\sigma(1,0)+1)b}\varphi_N^{[n]}\tilde{P}_{na-mb,-n'a+m'b}^{(N)}|\lambda\rangle_N & \text{otherwise} \end{cases} \tag{3.6.21}$$

It follows from the above

$$\sigma(P_{a,b})\hat{\psi}|\lambda\rangle_\infty = \sigma(P_{a,b})\hat{\psi}\,\varphi_N^{[1]}|\lambda\rangle_N = \sigma(P_{a,b})\varphi_N^{[n]}\tilde{\psi}|\lambda\rangle_N = \cdots = \hat{\psi}P_{a,b}\varphi_N^{[1]}|\lambda\rangle_N = \hat{\psi}P_{a,b}|\lambda\rangle_\infty,$$

here dots stands for the omitted steps involving the cases (straightforward to write down using (3.6.21), (3.6.16), and (3.6.19)). □

3.6.2 Action of the whole algebra

Theorem 3.6.1. *The following formulas determine an action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ on $\Lambda_{v,0}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$*

$$c \mapsto v^{-n}, \qquad c' \mapsto v^{-n'}, \tag{3.6.22a}$$

$$P_{0,-j} \mapsto q^j B_{-j}, \qquad P_{0,j} \mapsto \frac{q^j - 1}{v^{-2j}q^j - 1} v^{-jn} B_j, \tag{3.6.22b}$$

$$E(z) \mapsto u^{-\frac{1}{n}} q^{\frac{n+1}{2n}} \sum_{\alpha-\beta \equiv n' \bmod n} q^{-\frac{\alpha}{n}} z^{\frac{\beta-\alpha+n'}{n}} \hat{\Psi}_{(\alpha)}(qz)\hat{\Phi}^*_{(\beta)}(z), \tag{3.6.22c}$$

$$F(z) \mapsto u^{\frac{1}{n}} q^{\frac{n-1}{2n}} v^{-n'} \sum_{\alpha-\beta \equiv -n' \bmod n} q^{\frac{\beta}{n}} z^{\frac{\beta-\alpha-n'}{n}} \hat{\Phi}_{(\alpha)}(v^n z)\hat{\Psi}^*_{(\beta)}(qv^n z). \tag{3.6.22d}$$

for $q_1 = q$, $q_2 = v^{-2}$. The obtained representation is isomorphic to twisted Fock module $\mathcal{F}_u^\sigma$.

Proof. There is an action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ on $\Lambda_{v,0}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$ determined as follows. Recall that we have defined a map $\hat{\psi}$ from $\mathcal{F}_u^{\downarrow}$ to $\Lambda_{v,0}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right) = \mathcal{F}_u^{\swarrow}$. But there is an action of the whole $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ on $\mathcal{F}_u$, which coincides with $\mathcal{F}_u^{\downarrow}$ as a vector space. Hence, for any $X \in U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ and $w \in \Lambda_{v,0}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$ we can define $\rho_{tw}(X)w := \hat{\psi} \circ \rho_{\mathcal{F}_u}\left(\sigma^{-1}(X)\right) \circ \hat{\psi}^{-1}w$. The representation obtained is isomorphic to twisted Fock module $\mathcal{F}_u^\sigma$. In particular, formula (3.3.2) for $\tau = \sigma^{-1}$ implies $\rho_{tw}(c) = v^{-n}$ and $\rho(c') = v^{-n'}$. It remains to prove that the action ρ_{tw} is is given by (3.6.22b)–(3.6.22d).

Note, that now we have two actions of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ on $\Lambda_{v,0}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$. The first one comes from Proposition 3.5.4, let us denote it by ρ_+. The second one comes from the restriction of ρ_{tw} to $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$.

Lemma 3.6.5. *$\rho_+(P_{a,b}) = \rho_{tw}(P_{a,b})$ for $n'a + nb \leq 0$ and $b \geq 0$.*

Proof. Analogous to proof of Lemma 3.6.2. □

Lemma 3.6.6. *The actions ρ_+ and $\rho_{tw}\big|_{U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+}$ coincide.*

Proof. We use notation $|m\rangle = e_{-m} \wedge e_{-m+1} \wedge \dots$. We will simply write $P_{a,b} = \rho_+(P_{a,b}) = \rho_{tw}(P_{a,b})$ for $n'a + nb \leq 0$ and $b \geq 1$.

Any vector of $\Lambda_{v,0}^{\infty/2}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$ is a linear combination of vector $P_{a_1,b_1} \dots P_{a_t,b_t}|m\rangle$ for $n'a_i + nb_i < 0$ and $b_i \geq 0$. The following proposition is [BS12b, Lemma 5.6].

Proposition 3.6.4. *Algebra $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ has a basis $P_{k_1,l_1} \dots P_{k_t,l_t}$ for $\frac{l_1}{k_1} \leq \frac{l_2}{k_2} \leq \dots \leq \frac{l_t}{k_t}$.*

Hence action of $P_{k,l} \in U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ for $n'k + nl > 0$ is determined by commutation relations in $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^+$ and the conditions $P_{\tilde{k},\tilde{l}}|m\rangle = 0$ for $n'\tilde{k} + n\tilde{l} > 0$. □

Let us denote by ρ_- the action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)^-$, coming from Proposition 3.5.5. The following lemma is analogous to Lemma 3.6.6.

Lemma 3.6.7. *It holds $\rho_{tw}(P_{a,b}) = v^{-an'}\rho_-(P_{a,b})$.*

Proposition 3.5.4 imply that $\rho_{tw}(E(z)) = \rho_+(E(z))$ is given by (3.6.22c). Analogously, Proposition 3.5.5 imply that $\rho_{tw}(F(z)) = v^{-n'}\rho_-(F(z))$ is given by (3.6.22d). To find the action of $\rho_{tw}(P_{0,j})$, one can use either ρ_+ or ρ_-. □

3.7 Standard basis

As was already mentioned in the Introduction, one of the motivations of this paper is Gorsky-Neguţ conjecture on stable envelope bases in equivariant K-theory of Hilbert schemes of points on $\mathbb{C}^2$ [GN17]. Let us recall this conjecture.

We use notations n, n', m, m' as before, see Section 3.6.1. The Fock module $\mathcal{F}_u$ as a vector space can be identified with space of symmetric functions Λ using the correspondence $p_k \leftrightarrow P_{0,-k}$, for $k > 0$. Hence the twisted Fock module $\mathcal{F}_u^\sigma$ can be identified with Λ using correspondence $p_k \leftrightarrow P_{km,-km'}$, for $k > 0$.

There are several classical bases in the space Λ, e.g. Schur basis. Another important basis consists of *symmetric* Macdonald polynomials P_λ and renormalized ones M_λ, see [Neg16b, Sec. 2.4] and references therein. On the other hand, it was shown in Lemma 3.6.1 that the action of $P_{km,-km'}$ stabilize in $|\lambda\rangle_N$. Hence the vectors $|\lambda\rangle_\infty$ form basis in Λ. In order to identify this basis with stable envelope basis it should satisfy the following properties.

(i) $|\lambda\rangle_\infty$ are integral, i.e. they can be expanded in Schur basis with coefficients in $\mathbb{Z}[q^{\pm 1}, v^{\pm 1}]$.

(ii) $|\lambda\rangle_\infty$ are triangular with respect to dominance order "$<$" in M_λ basis

(iii) The coefficients $c_\lambda^\mu(q,v)$ in the expansion $|\lambda\rangle_\infty = \sum c_\lambda^\mu(q,v) M_\mu$ satisfy "window" condition.

We do not write precise form of window condition since we do not use it here.

Actually the conjecture in [GN17] was posed in a weaker form, namely it was stated there exists identification of $\Lambda_v^{\infty/2}(\mathbb{C}^n[Y^{\pm 1}])$ with space of symmetric functions Λ such that image of the basis $|\lambda\rangle$ satisfy properties (i)–(iii) and similarly for $\overline{|\lambda\rangle}$. In this form the conjecture was proven in [KS20b] using 3d mirror symmetry. Above we conjectured additionally that this identification comes from our Theorem 3.6.1.

We prove properties (i) and (ii) using analogues properties of the (nonsymmetric, finite) basis $e_\lambda = e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_N}$ shown in Section 3.2. The proofs are always consist of two steps: first we apply $\mathbf{S}_-$ and then take the limit $N \to \infty$. We believe that property (iii) can be also deduced from some analogue of window condition for basis e_λ.

In agreement with notations of Section 3.2, let us denote

$$e_{-\lambda-\rho} = e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1}. \tag{3.7.1}$$

When we want to stress number of variables we write $e_{-\lambda-\rho,N}$. Clearly we have $|\lambda\rangle_N = \mathbf{S}_- e_{-\lambda-\rho,N}$. Similarly, we use notations $A_{\lambda,N}$, $\tilde{E}_{\lambda,N}$ for monomial and Macdonald bases in space $(\mathbb{C}^n[Y^{\pm 1}])^{\otimes N}$ of functions on N variables, see formula (3.2.26) and Corollary 3.2.3 correspondingly.

Theorem 3.7.1. *The basis $|\lambda\rangle_\infty$ is triangular with respect to M_λ basis.*

Proof. It follows from the definition of $\tilde{E}_{\eta,N}$ (see Corollary 3.2.3) that we have the decomposition

$$e_{-\lambda-\rho,N} = \sum_{\eta \preceq -\lambda-\rho} \beta_{\lambda,\eta} \tilde{E}_{\eta,N}. \tag{3.7.2}$$

Note that here η is a composition, not a partition. Let us apply $\mathbf{S}_-$ to both sides. To calculate the action of $\mathbf{S}_-$ on RHS of (3.7.2) we need certain preparations.

It follows from the formulas (3.2.22) that

$$\Big(\sum\nolimits_{i=1}^N \mathbf{B}_i\Big)\mathbf{S}_-\tilde{E}_{\eta,N} = u_0 \cdot \ldots u_{N-1} q^{1-n}\Big(\sum\nolimits_{i=1}^N q^{\eta_i} v^{1_i^\eta}\Big)\mathbf{S}_-\tilde{E}_{\eta,N}, \tag{3.7.3}$$

where we used commutativity of $\sum \mathbf{B}_i$ and finite Hecke algebra H. This formula does not depend on order of parts η_i, hence without loss of generality we can assume that $\eta_1 \le \eta_2 \le \cdots \le \eta_N$. Then the eigenvalue takes the form $\sim \sum q^{\eta_i} v^{-2i}$, where $\sim$ stands for factor that does not depend on η.

On the other hand we can use Theorem 3.2.4 and identify $(\mathbb{C}^n[Y^{\pm 1}])^{\otimes N}$ with $(\mathbb{C}[\mathbf{A}^{\pm 1}])^{\otimes N}$. It is known (see e.g. [Kir97, Sec. 7]) that all eigenvectors of $\sum \mathbf{B}_i$ in $\mathbf{S}_-(\mathbb{C}[\mathbf{A}^{\pm 1}])^{\otimes N}$ has the form $P_{\mu,N}(\mathbf{A}^{-1}; q, qv^{-2})\mathcal{A}$. Here $\mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_N)$ is a partition (possibly with negative parts), $P_{\mu,N}(\mathbf{A}^{-1}; q, qv^{-2})$ is a *symmetric* Macdonald polynomial on N variables $\mathbf{A}_1^{-1}, \ldots, \mathbf{A}_N^{-1}$ and $\mathcal{A} = \frac{1}{[N]_v}\prod_{i<j}(vA_i - v^{-1}A_j)$. The corresponding eigenvalue has the form $\sim \sum_i q^{-\mu_i}(q^{-1}v^2)^{-i}$ where $\sim$ stands for factor that does not depend on μ.

Comparing two formulas for eigenvalue, we see that composition η should be related to partition μ by the formula $\mu_i = -\eta_i + i - 1$ and in this case $\mathbf{S}_- E_{\eta,N} \sim P_{\mu,N}(\mathbf{A}^{-1}; q, qv^{-2})\mathcal{A}$, here $\sim$ is some constant depending on η, v, q. In particular if some parts of η are equal then there is no such partition μ and $\mathbf{S}_- E_{\eta,N} = 0$. For the expansion (3.7.2), it follows from $\nu \prec -\lambda-\rho$ that parts of μ are nonegative and $\mu \le \lambda$. Therefore we get

$$|\lambda\rangle_N = \sum_{\mu \le \lambda} \tilde{\beta}_{\lambda,\mu} P_{\mu,N}(\mathbf{A}^{-1})\mathcal{A}. \tag{3.7.4}$$

Taking the limit $N \to \infty$, we get the theorem. □

Theorem 3.7.2. *The vectors $|\lambda\rangle_\infty$ are integral.*

Proof. It follows from Theorem 3.2.3 that we have expansion

$$e_{-\lambda-\rho,N} = \sum_{\eta \preceq -\lambda-\rho} \tilde{\alpha}_{\lambda,\eta} A_{\eta,N}, \tag{3.7.5}$$

where $\tilde{\alpha}_{\lambda,\eta} \in \mathbb{Z}[q^{\pm 1}, v^{\pm 1}]$. We apply $\mathbf{S}_-$ to both sides. It remains to show that the vectors $\mathbf{S}_- A_{\eta,N}$ are integral.

Introduce usual (not v-deformed) antisymmetrizer $\mathbf{S}_-^{v=1} = \frac{1}{N!}\sum(-1)^{l(w)} w^{\mathbf{A}}$, here summation goes over all permuations and $w^{\mathbf{A}}$ permutes the operators $\mathbf{A}_i$. It was shown in [Kir97, Th. 8.2] that $\operatorname{Ker}\mathbf{S}_-^{v=1} = \operatorname{Ker}\mathbf{S}_-$. Hence the vector $\mathbf{S}_- A_{\eta,N}$ is skew symmetric in η_i. Therefore it is sufficient to consider the case $\eta_1 < \eta_2 < \cdots < \eta_N$.

Introduce partition μ by the formula $\mu_i = -\eta_i + i - 1$. We have $\mathbf{S}_- A_{\eta,N} = f_\mu(\mathbf{A}^{-1})\mathcal{A}$ for some symmetric polynomial since it belongs to the image of $\mathbf{S}_-$, see [Kir97, proof of Th. 7.1]. We claim that $f_\mu(\mathbf{A}^{-1}) = s_\mu(\mathbf{A}^{-1})$. This is obvious for $v = 1$. For generic v it is easy to check that $f_\varnothing = 1$ and multiplication of f_μ by $\sum_i \mathbf{A}_i^{-k}$ is given by formulas which do not depend on v (Murnaghan-Nakayama rule). Evidently, f_μ is determined by the formulas.

To finish the proof we take the limit $N \to \infty$. □

Remark 3.7.1. Similarly using (3.4.8) one can show that costandard basis $\overline{|\lambda\rangle}$ satisfies properties (i) and (ii). This is also in agreement with conjectures of [GN17].

3.8 Quantum affine algebra and its vertex operators

In this paper, we have used the space $\Lambda_{v,m}^{\infty/2}\big(\mathbb{C}^n[Y^{\pm 1}]\big)$ and the vertex operators $\Phi(z)$, $\Psi(z)$, $\Phi^*(z)$, and $\Psi^*(z)$. The space $\Lambda_{v,m}^{\infty/2}\big(\mathbb{C}^n[Y^{\pm 1}]\big)$ is known as an integrable level-one representation of quantum affine $U_v(\widehat{\mathfrak{gl}}_n)$ [KMS95, LT00]. The vertex operators can be defined via intertwining properties.

Integrable level-one representations and the vertex operators have their counterpart for $U_v(\widehat{\mathfrak{sl}}_n)$. Below we will study connection between the $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ versions.

We will consider only the vertex operator $\Phi^*(z)$. The situation for other operators is analogous. We will need the results concerning $\Phi^*(z)$ for the proof of Proposition 3.4.8.

3.8.1 Action of quantum affine algebra

Let $\alpha_0, \ldots, \alpha_{n-1}$ be simple positive roots of $\widehat{\mathfrak{sl}}_n$. We will use standard scale product (α_i, α_j), and Cartan matrix $\langle\alpha_i, \alpha_j\rangle = 2\,(\alpha_i, \alpha_j)\,/\,(\alpha_j, \alpha_j)$. Note that for $\hat{\mathfrak{sl}}_n$ we have $\langle\alpha_i, \alpha_j\rangle = (\alpha_i, \alpha_j)$. Algebra $U_v(\widehat{\mathfrak{sl}}_n)$ is generated by $\mathtt{E}_i$, $\mathtt{K}_i$ and $\mathtt{F}_i$ for $i = 0, 1, \ldots, n-1$. The relations are

$$\mathtt{K}_i\mathtt{K}_j = \mathtt{K}_j\mathtt{K}_i, \qquad \mathtt{K}_i\mathtt{E}_j\mathtt{K}_i^{-1} = v^{(\alpha_i,\alpha_j)}\mathtt{E}_j, \qquad \mathtt{K}_i\mathtt{F}_j\mathtt{K}_i^{-1} = v^{-(\alpha_i,\alpha_j)}\mathtt{F}_j, \tag{3.8.1}$$

$$[\mathtt{E}_i, \mathtt{F}_j] = \delta_{i,j}\frac{\mathtt{K}_i - \mathtt{K}_i^{-1}}{v - v^{-1}}, \tag{3.8.2}$$

$$\sum_{k=0}^{b_{ij}}(-1)^k \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_v \mathtt{E}_i^k\mathtt{E}_j\mathtt{E}_i^{b_{ij}-k} = 0, \qquad \sum_{k=0}^{b_{ij}}(-1)^k \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_v \mathtt{E}_i^k\mathtt{E}_j\mathtt{E}_i^{b_{ij}-k} = 0. \tag{3.8.3}$$

here $b_{ij} = 1 - \langle\alpha_i, \alpha_j\rangle$ and $\begin{bmatrix} b_{ij} \\ k \end{bmatrix}_v = [b_{ij}]_v!/\big([k]_v![b_{ij}-k]_v!\big)$. There is an action of $U_v(\widehat{\mathfrak{sl}}_n)$ on $\mathbb{C}^n[Y^{\pm 1}]$ determined as follows

$$\mathtt{E}_i e_j = \delta_{i\equiv j} e_{j+1} \qquad \mathtt{F}_i e_j = \delta_{i\equiv j-1} e_{j-1} \qquad \mathtt{K}_i e_j = v^{\delta_{i\equiv j-1} - \delta_{i\equiv j}} e_j \tag{3.8.4}$$

Using the comultiplication

$$\Delta(\mathtt{K}_i) = \mathtt{K}_i \otimes \mathtt{K}_i \qquad \Delta(\mathtt{E}_i) = \mathtt{E}_i \otimes 1 + \mathtt{K}_i^{-1} \otimes \mathtt{E}_i \qquad \Delta(\mathtt{F}_i) = \mathtt{F}_i \otimes \mathtt{K}_i + 1 \otimes \mathtt{F}_i \tag{3.8.5}$$

we can define action of $U_v(\widehat{\mathfrak{sl}}_n)$ on $\left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$.

Proposition 3.8.1. *Action of (affine) Hecke algebra commutes with $U_v(\widehat{\mathfrak{sl}}_n)$.*

Hence we have obtained an action $U_v(\widehat{\mathfrak{sl}}_n) \curvearrowright \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$ [KMS95, LT00].

The limit Let us use the inductive system (3.4.11). Let $\mathtt{E}_i^{(N)}, \mathtt{K}_i^{(N)}, F_i^{(N)} \curvearrowright \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$ be the operators, coming from the defined above action $U_v(\widehat{\mathfrak{sl}}_n) \curvearrowright \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$. Consider a number $\mathtt{r} = 0, 1, \ldots, n-1$ such that $\mathtt{r} \equiv N - 1 - m \pmod n$. Let us define the following operators

$$\tilde{\mathtt{E}}_i^{(N)} = \mathtt{E}_i^{(N)} \qquad \tilde{\mathtt{K}}_i^{(N)} = v^{\delta_{i,\mathtt{r}}} \mathtt{K}_i^{(N)} \qquad \tilde{\mathtt{F}}_i^{(N)} = v^{\delta_{i,\mathtt{r}}} \mathtt{F}_i^{(N)}. \tag{3.8.6}$$

Below we will need the following versions of Definitions 3.4.1 and 3.4.2.

Definition 3.8.1. Action of operators $A^{(N)} \colon \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \to \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N+\Delta}$ *1-stabilizes* if for any $w \in \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes k}$ there is M such that for any $N > M$ we have

$$\varphi^{(m)}_{N+\Delta+1, N+\Delta} \circ A^{(N)} \circ \varphi^{(m)}_{N,k}(w) = A^{(N+1)} \circ \varphi^{(m)}_{N+1,k}(w) \tag{3.8.7}$$

Definition 3.8.2. Action of operators $A^{(N)} \colon \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \to \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N+\Delta}$ *weakly 1-stabilizes* if for any $w \in \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes k}$ there is M such that for any $N > M$ we have

$$\varphi^{(m)}_{N+\Delta} \circ A^{(N)} \circ \varphi^{(m)}_{N,k}(w) = \varphi^{(m)}_{N+\Delta+1} \circ A^{(N+1)} \circ \varphi^{(m)}_{N+1,k}(w) \tag{3.8.8}$$

Proposition 3.8.2. *The operators $\tilde{\Phi}_k^*$ 1-stabilize.*

Proposition 3.8.3. *The operators $\tilde{\mathtt{F}}_i^{(N)}$ and $\tilde{\mathtt{K}}_i^{(N)}$ 1-stabilize. The operators $\tilde{\mathtt{E}}_i^{(N)}$ weakly 1-stabilize. Moreover, for $i \neq \mathtt{r}$ the operator $\mathtt{E}_i^{(N)}$ stabilizes. More explicitly, for any $w \in \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes k}$ and sufficiently large $N \not\equiv m + i + 1 \pmod n$ we have*

$$\varphi^{(m)}_{N+n,N} \circ \mathtt{E}_i^{(N)} \circ \varphi^{(m)}_{N,k}(w) = \mathtt{E}_i^{(N+n)} \circ \varphi^{(m)}_{N+1,k}(w) \tag{3.8.9}$$

Denote the induced operators by $\hat{\mathtt{E}}_i$, $\hat{\mathtt{K}}_i$, and $\hat{\mathtt{F}}_i$.

Proposition 3.8.4 ([KMS95, LT00])**.** *The formulas $\mathtt{E}_i \mapsto \hat{\mathtt{E}}_i$, $\mathtt{K}_i \mapsto \hat{\mathtt{K}}_i$, $\mathtt{F}_i \mapsto \hat{\mathtt{F}}_i$ determine an action of $U_v(\widehat{\mathfrak{sl}}_n)$ on $\Lambda^{\infty/2}_{v,m}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$.*

From $\mathfrak{sl}_n$ to $\mathfrak{gl}_n$ Let $U_v(\mathsf{Heis})$ be the algebra generated by B_k for $k \in \mathbb{Z}$ and central $c^{\pm 1}$ with the relation

$$[B_k, B_l] = k \frac{c^{-2k} - 1}{v^{2k} - 1} \delta_{k+l,0} \tag{3.8.10}$$

Remark 3.8.1. We abuse notation since B_k was defined as an operator on $\Lambda^{\infty/2}_{v,m}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$ and c is an element of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$.

Let $U_v(\widehat{\mathfrak{gl}}_n) = U_v(\widehat{\mathfrak{sl}}_n) \otimes U_v(\mathsf{Heis})$. The algebra $U_v(\widehat{\mathfrak{gl}}_n)$ acts on $\Lambda^{\infty/2}_{v,m}\left(\mathbb{C}^n[Y^{\pm 1}]\right)$ as follows. Action of $U_v(\widehat{\mathfrak{sl}}_n)$ comes from Proposition 3.8.4. The generators B_k act as the operators with the same name defined in Proposition 3.4.6. The central element c acts as multiplication by v^{-n}. Denote the obtained representation by F_m.

Proposition 3.8.5. *The obtained representation F_m is irreducible integrable level-one representation of $U_v(\widehat{\mathfrak{gl}}_n)$. There are n non-isomorphic classes of such representations $F_0, \dots, F_{n-1}$. The representations F_m and F_{m+nk} are isomphic for any $k \in \mathbb{Z}$.*

Let $F_0^{sl}, F_1^{sl}, \dots, F_{n-1}^{sl}$ be the irreducible integrable level-one representation of $U_v(\widehat{\mathfrak{sl}}_n)$. Let F^{H} be Fock module of $U_v(\mathsf{Heis})$ for $c = v^{-n}$.

Proposition 3.8.6. *The representations F_i is isomorphic to $F_i^{sl} \otimes F^{\mathsf{H}}$ as representations of $U_v(\widehat{\mathfrak{gl}}_n) = U_v(\widehat{\mathfrak{sl}}_n) \otimes U_v(\mathsf{Heis})$ for $i = 0, 1, \dots, n-1$.*

3.8.2 Vertex operators

Below we will study intertwining property of $\hat{\Phi}_k^*$. Its analogue defines vertex operators for $\mathfrak{sl}_n$ [FR92]. In this subsection we will recall basic properties of the vertex operators for $\mathfrak{sl}_n$ and start to study connection between vertex operators of $\mathfrak{sl}_n$ and $\hat{\Phi}_k$. The connection will be made more precise in the next subsection.

Intertwining property Let us define an operator by

$$\hat{\Phi}^* \colon \Lambda^{\infty/2}_{v,m}\big(\mathbb{C}^n[Y^{\pm 1}]\big) \to \mathbb{C}^n[Y^{\pm 1}] \hat{\otimes} \Lambda^{\infty/2}_{v,m-1}\big(\mathbb{C}^n[Y^{\pm 1}]\big) \tag{3.8.11}$$

$$\hat{\Phi}^* w = \sum_{k \in \mathbb{Z}} e_{-k} \otimes \hat{\Phi}_k^* w. \tag{3.8.12}$$

Proposition 3.8.7. $\hat{\Phi}^*$ *is an $U_v(\widehat{\mathfrak{sl}}_n)$-intertwiner*

Evidently, Proposition 3.8.7 is equivalent to the following proposition.

Proposition 3.8.8. *The following relations hold*

$$\hat{\Phi}_k^* \hat{\mathtt{K}}_i = v^{\delta_{i+k\equiv -1} - \delta_{i+k\equiv 0}} \hat{\mathtt{K}}_i \hat{\Phi}_k^* \tag{3.8.13}$$

$$\hat{\Phi}_k^* \hat{\mathtt{E}}_i = \delta_{i+k\equiv -1} \hat{\Phi}_{k+1}^* + v^{\delta_{i+k\equiv 0} - \delta_{i+k\equiv -1}} \hat{\mathtt{E}}_i \hat{\Phi}_k^* \tag{3.8.14}$$

$$\hat{\Phi}_k^* \hat{\mathtt{F}}_i = \delta_{i+k\equiv 0} \hat{\mathtt{K}}_i \hat{\Phi}_{k-1}^* + \hat{\mathtt{F}}_i \hat{\Phi}_k^* \tag{3.8.15}$$

Proof. Analogously to (3.8.12), consider operator

$$\Phi^* \colon \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \to \left(\mathbb{C}^n[Y^{\pm 1}]\right) \otimes \mathbf{S}_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N-1} \tag{3.8.16}$$

$$\Phi^* w = \sum_{k \in \mathbb{Z}} e_{-k} \otimes \Phi_k^* w \tag{3.8.17}$$

Evidently, the operator is an intertwiner. Equivalently, Φ_k^* satisfy the counterparts of (3.8.13)–(3.8.15)

$$\Phi_k^* \mathtt{K}_i^{(N)} = v^{\delta_{i+k\equiv -1} - \delta_{i+k\equiv 0}} \mathtt{K}_i^{(N-1)} \Phi_k^* \tag{3.8.18}$$

$$\Phi_k^* \mathtt{E}_i^{(N)} = \delta_{i+k\equiv -1} \Phi_{k+1}^* + v^{\delta_{i+k\equiv 0} - \delta_{i+k\equiv -1}} \mathtt{E}_i^{(N-1)} \Phi_k^* \tag{3.8.19}$$

$$\Phi_k^* \mathtt{F}_i^{(N)} = \delta_{i+k\equiv 0} \mathtt{K}_i^{(N-1)} \Phi_{k-1}^* + \mathtt{F}_i^{(N-1)} \Phi_k^* \tag{3.8.20}$$

Note that in the relations above we can replace $\mathtt{K}_i^{(N)}, \mathtt{E}_i^{(N)}, \mathtt{F}_j^{(N)}$ by $\tilde{\mathtt{K}}_i^{(N)}, \tilde{\mathtt{E}}_i^{(N)}, \tilde{\mathtt{F}}_j^{(N)}$ correspondingly. Since the operators $\tilde{\mathtt{K}}_i^{(N)}$, $\tilde{\mathtt{E}}_i^{(N)}$, and $\tilde{\mathtt{F}}_j^{(N)}$ stabilize for $j \neq \mathtt{r}$, the corresponding relation hold for $\hat{\Phi}_k^*$. It remains to check

$$\hat{\Phi}_k^* \hat{\mathtt{E}}_{\mathtt{r}} = \delta_{\mathtt{r}+k\equiv -1} \hat{\Phi}_{k+1}^* + v^{\delta_{\mathtt{r}+k\equiv 0} - \delta_{\mathtt{r}+k\equiv -1}} \hat{\mathtt{E}}_{\mathtt{r}} \hat{\Phi}_k^* \tag{3.8.21}$$

Notice that $E_{\mathtt{r}}^{(N+1)}$ stabilizes. Hence (3.8.21) follows from

$$\Phi_k^* \mathtt{E}_{\mathtt{r}}^{(N+1)} = \delta_{\mathtt{r}+k\equiv -1} \Phi_{k+1}^* + v^{\delta_{\mathtt{r}+k\equiv 0} - \delta_{\mathtt{r}+k\equiv -1}} \mathtt{E}_{\mathtt{r}}^{(N)} \Phi_k^* \tag{3.8.22}$$

□

Vertex operators for $U_v(\widehat{\mathfrak{sl}}_n)$ Abusing notation, we will denote the highest vector of F_i^{sl} by $|i\rangle$. Let us also define F_i^{sl} for any $i \in \mathbb{Z}$ by $F_i = F_i^{sl} \otimes F^{\mathsf{H}}$.

Vertex operators for $U_v(\widehat{\mathfrak{sl}}_n)$ are defined as the following intertwiners

$$\hat{\Phi}^{*,sl} \colon F_i^{sl} \to \left(\mathbb{C}^n[Y^{\pm 1}]\right) \hat{\otimes} F_{i-1}^{sl} \tag{3.8.23}$$

$$\hat{\Phi}^{*,sl}|i\rangle = e_{-i} \otimes |i-1\rangle + \sum_{k<i} e_{-k} \otimes w_k \tag{3.8.24}$$

for certain vectors $w_k \in F_{i-1}$.

Proposition 3.8.9 ([FR92])**.** *There exists unique operator $\hat{\Phi}^{*,sl}$ determined by* (3.8.23)–(3.8.24)*.*

Also, we define operators $\hat{\Phi}_k^{*,sl}$ and currents $\hat{\Phi}_{(\alpha)}^{*,sl}(z)$ by

$$\hat{\Phi}^{*,sl} w = \sum_{k\in\mathbb{Z}} e_{-k} \otimes \hat{\Phi}_k^{*,sl} w \qquad\qquad \hat{\Phi}_{(\alpha)}^{*,sl}(z) = \sum_{k\in\mathbb{Z}} \Phi_{-\alpha+nk}^{*,sl} z^{-k} \tag{3.8.25}$$

Let us consider *principal grading* on F_i^{sl} given by $\deg|i\rangle = -\frac{i(i+1)}{2}$, $\deg \mathsf{E}_i = 1$, $\deg \mathsf{F}_i = -1$. Note that $\deg \hat{\Phi}_k^{*,sl} = k$.

Lemma 3.8.1. *For any intertwiner $\phi^{*,sl} = \sum e_{-k} \otimes \phi_k^{*,sl} \colon F_i^{sl} \to \left(\mathbb{C}^n[Y^{\pm 1}]\right) \hat{\otimes} F_{i-1}^{sl}$ such that $\deg \phi_k^{*,sl} = k + \Delta$, we have $\phi_k^{*,sl} = \gamma \hat{\Phi}_{k+\Delta}^{*,sl}$ for certain $\gamma \in \mathbb{C}$*

Proof. Note that $\deg \phi_{i-\Delta}^{*,sl}|i\rangle = \deg|i-1\rangle$. Since the subspace of degree $\deg|i-1\rangle$ is one-dimensional, we have $\phi_{i-\Delta}^{*,sl}|i\rangle = \gamma|i-1\rangle$ for certain $\gamma \in \mathbb{C}$. Proposition 3.8.9 implies $\sum e_{-k} \otimes \phi_{k-\Delta}^{*,sl} = \gamma \hat{\Phi}^{*,sl}$. □

Proposition 3.8.10. *There exist operators $\Xi_d \curvearrowright F^{\mathsf{H}}$ such that for $\Xi(z) = \sum_{d\in\mathbb{Z}} \Xi_d z^{-d}$ we have*

$$\hat{\Phi}^*_{(\alpha)}(z) = \hat{\Phi}_{(\alpha)}^{*,\mathfrak{sl}_n}(z) \otimes \Xi(z) \tag{3.8.26}$$

Proof. We can extend the grading from F_i^{sl} to F_i by $\deg B_k = kn$. Note that $\deg \hat{\Phi}_k = k$. We can present $\hat{\Phi}_k = \sum_{d,\nu} \phi_{k,d,\nu}^{*,sl} \otimes \Xi_{d,\nu}$ for linear independent operators $\Xi_{d,\nu}$ with $\deg \Xi_{d,\nu} = nd$ (e.g. take $\Xi_{d,\nu}$ to be matrix units for a homogeneous basis of F^{H}). Proposition 3.8.7 and Lemma 3.8.1 imply that $\phi_{k,d,\nu}^{*,sl} = \gamma_{d,\nu} \hat{\Phi}_{k-nd}^{*,sl}$ for certain $\gamma_{d,\nu} \in \mathbb{C}$. Hence $\hat{\Phi}_k = \sum_d \hat{\Phi}_{k-nd}^{*,sl} \otimes (\sum_\nu \gamma_{d,\nu} \Xi_{d,\nu})$. □

Bosonization The operator $\hat{\Phi}_{(n-1)}^{*,sl}(z)$ was calculated in [Koy94, Thm. 3.4]. Note that the parameter q used in the *loc. cit.* corresponds to our parameter $v = v^{-1}$. To write the answer, we recall notation of *loc. cit.*.

Let $\bar{Q}$ and $\bar{P}$ be root and weight lattices for $\mathfrak{sl}_n$ correspondingly. We use notation e^β for an element of group algebra $\mathbb{C}[\bar{P}]$, corresponding to $\beta \in \bar{P}$. Denote the fundamental weights of $\mathfrak{sl}_n$ by $\bar{\Lambda}_1, \ldots, \bar{\Lambda}_{n-1}$. There is Heisenberg algebra in $U_v(\widehat{\mathfrak{sl}}_n)$ generated by $a_j(k)$ for $k \in \mathbb{Z}$ and $j = 1, \ldots, n-1$. Let F^a be Fock module for the Heisenberg algebra. Then F_i can be naturally identified with $F^a \otimes \mathbb{C}[\bar{Q}] e^{\bar{\Lambda}_i}$, and the action of $U_v(\widehat{\mathfrak{sl}}_n)$ can be constructed explicitly, see [FJ88] or [Koy94, Sect. 2.4].

For $\alpha \in \bar{P}$, let us introduce operator $\partial_\alpha (w \otimes \beta) = (\alpha, \beta) w \otimes \beta$. There exists subalgebra $a_1^*(k)$ in the algebra generated by $a_j(k)$. In the representation F_i, the operators satisfy

$$[a_1^*(k), a_1^*(-k)] = -\frac{[(n-1)k]_v}{k[nk]_v} \tag{3.8.27}$$

Proposition 3.8.11 ([Koy94]). *Vertex operator $\hat{\Phi}^{*,sl}_{(n-1)}(z)\colon F_i \to F_{i-1}$ is given by the following formula*

$$\hat{\Phi}^{*,sl}_{(n-1)}(z) = \exp\left(-\sum_{k=1}^{\infty} a_1^*(-k)v^{-\frac{1}{2}k}z^k\right)\exp\left(-\sum_{k=1}^{\infty} a_1^*(k)v^{\frac{3}{2}k}z^{-k}\right) \times e^{-\bar{\Lambda}_1}\left((-1)^{n-1}v^{-1}z\right)^{-\partial_{\bar{\Lambda}_1}+\frac{n-i-1}{n}} v^i(-1)^{in+\frac{1}{2}i(i+1)} \quad (3.8.28)$$

Normal ordered product $:\hat{\Phi}^{*,sl}_{(n-1)}(z_1)\hat{\Phi}^{*,sl}_{(n-1)}(z_2):$ is an operator from F_i to F_{i-2} defined by the following formula

$$:\hat{\Phi}^{*,sl}_{(n-1)}(z_1)\hat{\Phi}^{*,sl}_{(n-1)}(z_2):= \exp\left(-\sum_{k=1}^{\infty} a_1^*(-k)v^{-\frac{1}{2}k}(z_1^k+z_2^k)\right)\exp\left(-\sum_{k=1}^{\infty} a_1^*(k)v^{\frac{3}{2}k}(z_1^{-k}+z_2^{-k})\right) \times e^{-2\bar{\Lambda}_1}\prod_{j=1,2}\left((-1)^{n-1}v^{-1}z_j\right)^{-\partial_{\bar{\Lambda}_1}+(n-i_j-1)/n} v^{i_j}(-1)^{i_jn+\frac{1}{2}i_j(i_j+1)}, \quad (3.8.29)$$

here $i_j = i+2-j$. Note that the normal ordering is not symmetric, namely

$$z_1^{-\frac{1}{n}} :\hat{\Phi}^{*,sl}_{(n-1)}(z_1)\hat{\Phi}^{*,sl}_{(n-1)}(z_2):= z_2^{-\frac{1}{n}} :\hat{\Phi}^{*,sl}_{(n-1)}(z_2)\hat{\Phi}^{*,sl}_{(n-1)}(z_1): . \quad (3.8.30)$$

The following relations can be checked directly

$$\hat{\Phi}^{*,sl}_{(n-1)}(z_1)\hat{\Phi}^{*,sl}_{(n-1)}(z_2) = \left((-1)^{n-1}v^{-1}z_1\right)^{\frac{n-1}{n}}\frac{(v^2z_2/z_1, v^{2n})_\infty}{(v^{2n}z_2/z_1, v^{2n})_\infty} :\hat{\Phi}^{*,sl}_{(n-1)}(z_1)\hat{\Phi}^{*,sl}_{(n-1)}(z_2): . \quad (3.8.31)$$

3.8.3 Factorization of the vertex operator

We continue to study connection between vertex operators for $\mathfrak{sl}_n$ and $\hat{\Phi}^*_{(\alpha)}(z)$. This subsection is devoted to a proof of Theorem 3.8.1. The theorem is used for the proof of Proposition 3.4.8.

Theorem 3.8.1. *The following holds*

$$\hat{\Phi}^*_{(\alpha)}(z) = \hat{\Phi}^{*,\mathfrak{sl}_n}_{(\alpha)}(z)\otimes\exp\left(-\sum_{j>0}\frac{v^{2jn}}{j[n]^+_{v^j}}B_{-j}z^j\right)\exp\left(\sum_{j>0}\frac{1}{j[n]^+_{v^j}}B_jz^{-j}\right). \quad (3.8.32)$$

To prove the theorem we need certain preparations. Let us define

$$\Phi^*_{(\alpha)}(z) = \sum_{k\in\mathbb{Z}}\Phi^*_{-\alpha+nk}z^{-k} \quad (3.8.33)$$

Lemma 3.8.2. *The following relation holds*

$$(v^2z_1 - z_2)\Phi^*_{(\alpha)}(z_1)\Phi^*_{(\alpha)}(z_2) = (v^2z_2 - z_1)\Phi^*_{(\alpha)}(z_2)\Phi^*_{(\alpha)}(z_1) \quad (3.8.34)$$

Proof. Consider operator

$$(1\otimes\Phi^*)\circ\Phi^*\colon \mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N}\to\left(\mathbb{C}^n[Y^{\pm1}]\right)\otimes\left(\mathbb{C}^n[Y^{\pm1}]\right)\otimes\mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N-2} \quad (3.8.35)$$

$$(1\otimes\Phi^*)\circ\Phi^*w = \sum_{k,l\in\mathbb{Z}} e_l\otimes e_k\otimes\Phi^*_{-k}\Phi^*_{-l}w \quad (3.8.36)$$

Also, consider operator

$$T_{12}\curvearrowright\left(\mathbb{C}^n[Y^{\pm1}]\right)\otimes\left(\mathbb{C}^n[Y^{\pm1}]\right)\otimes\mathbf{S}_-\left(\mathbb{C}^n[Y^{\pm1}]\right)^{\otimes N-2} \quad (3.8.37)$$

induced from action of T on first two tensor multiples. Recall that T is given by (3.2.6)–(3.2.9). The basic property of anti-symmetrizer (3.3.10) implies

$$T_{12} \circ (1 \otimes \Phi^*) \circ \Phi^* = -v^{-1} (1 \otimes \Phi^*) \circ \Phi^*. \tag{3.8.38}$$

Let $l > k$ and $l \equiv k \bmod n$. Consider in (3.8.38) the coefficients in front of $e_l \otimes e_k$ and $e_{l+n} \otimes e_{k-n}$

$$v\Phi^*_{-l}\Phi^*_{-k} - (v - v^{-1}) \sum_{j=1}^{\infty} \Phi^*_{-k+nj}\Phi^*_{-l-nj} + (v - v^{-1}) \sum_{j=1}^{\infty} \Phi^*_{-l-nj}\Phi^*_{-k+nj} = -v^{-1}\Phi^*_{-k}\Phi^*_{-l} \tag{3.8.39}$$

$$v\Phi^*_{-l-n}\Phi^*_{-k+n} - (v - v^{-1}) \sum_{j=2}^{\infty} \Phi^*_{-k+nj}\Phi^*_{-l-nj} + (v - v^{-1}) \sum_{j=2}^{\infty} \Phi^*_{-l-nj}\Phi^*_{-k+nj} = -v^{-1}\Phi^*_{-k+n}\Phi^*_{-l-n} \tag{3.8.40}$$

Hence

$$\begin{aligned} v\left(\Phi^*_{-l}\Phi^*_{-k} - \Phi^*_{-l-n}\Phi^*_{-k+n}\right) - (v - v^{-1})\Phi^*_{-k+n}\Phi^*_{-l-n} + (v - v^{-1})\Phi^*_{-l-n}\Phi^*_{-k+n} \\ = -v^{-1}\left(\Phi^*_{-k}\Phi^*_{-l} - \Phi^*_{-k+n}\Phi^*_{-l-n}\right) \end{aligned} \tag{3.8.41}$$

Equivalently

$$v\Phi^*_{-l}\Phi^*_{-k} - v^{-1}\Phi^*_{-l-n}\Phi^*_{-k+n} = -v^{-1}\Phi^*_{-k}\Phi^*_{-l} + v\Phi^*_{-k+n}\Phi^*_{-l-n} \tag{3.8.42}$$

Substituting $l - n$ instead of l and multiplying by v, we obtain

$$v^2\Phi^*_{-l+n}\Phi^*_{-k} - \Phi^*_{-l}\Phi^*_{-k+n} = v^2\Phi^*_{-k+n}\Phi^*_{-l} - \Phi^*_{-k}\Phi^*_{-l+n} \tag{3.8.43}$$

To finish the proof we notice that (3.8.43) is symmetric on l and k. □

Proposition 3.8.12. *The following holds*

$$(v^2 z_1 - z_2)\hat{\Phi}^*_{(\alpha)}(z_1)\hat{\Phi}^*_{(\alpha)}(z_2) = (v^2 z_2 - z_1)\hat{\Phi}^*_{(\alpha)}(z_2)\hat{\Phi}^*_{(\alpha)}(z_1) \tag{3.8.44}$$

Proof. Follows from Lemma 3.8.2 since the operators Φ^*_k stabilize. □

Remark 3.8.2. Proposition 3.8.12 can be generalized. Namely, one can write interchanging relation for $\Phi^*_\alpha(z_1)$ and $\Phi^*_\beta(z_2)$. The result is *R-matrix relation* [DO94, eq.(2.17)] [JM95, eq.(6.31)]. We will neither formulate nor use the relation.

Proof of Theorem 3.8.1. Let us substitute (3.8.26) to (3.8.44) for $\alpha = n - 1$. Using relation (3.8.31), we obtain

$$\begin{aligned} (v^2 z_1 - z_2) z_1^{\frac{n-1}{n}} \frac{(v^2 z_2/z_1, v^{2n})_\infty}{(v^{2n} z_2/z_1, v^{2n})_\infty} :\hat{\Phi}^{*,sl}_{(n-1)}(z_1)\hat{\Phi}^{*,sl}_{(n-1)}(z_2): \otimes \Xi(z_1)\Xi(z_2) \\ = (v^2 z_2 - z_1) z_2^{\frac{n-1}{n}} \frac{(v^2 z_1/z_2, v^{2n})_\infty}{(v^{2n} z_1/z_2, v^{2n})_\infty} :\hat{\Phi}^{*,sl}_{(n-1)}(z_2)\hat{\Phi}^{*,sl}_{(n-1)}(z_1): \otimes \Xi(z_2)\Xi(z_1) \end{aligned} \tag{3.8.45}$$

Using (3.8.30), one can see that

$$(v^2 z_1 - z_2) z_1 \frac{(v^2 z_2/z_1, v^{2n})_\infty}{(v^{2n} z_2/z_1, v^{2n})_\infty} \Xi(z_1)\Xi(z_2) = (v^2 z_2 - z_1) z_2 \frac{(v^2 z_1/z_2, v^{2n})_\infty}{(v^{2n} z_1/z_2, v^{2n})_\infty} \Xi(z_2)\Xi(z_1) \tag{3.8.46}$$

The relation $[B_{-j}, \hat{\Phi}^*_k] = -\hat{\Phi}^*_{k-nj}$ implies

$$\Xi(z) = \Xi_-(z) \exp\left(\sum_{j>0} \frac{1}{j[n]^+_{v^j}} B_j z_1^{-j}\right), \tag{3.8.47}$$

here $\Xi_-(z)$ is a formal power series, the coefficients are operators on F^{H}, and $[B_{-j}, \Xi_-(z)] = 0$. Equivalently, $\Xi_-(z) = \sum_\mu \alpha_{\mu_1,\dots,\mu_j} B_{-\mu_1} \dots B_{-\mu_j} z^{|\mu|}$. Denote

$$\Xi_- \left[B_{-k} + z_1^{-k}\right](z_2) = \sum_\mu \alpha_{\mu_1,\dots,\mu_j} \left(B_{-\mu_1} + z_1^{-\mu_1}\right) \dots \left(B_{-\mu_j} + z_1^{-\mu_j}\right) z_2^{|\mu|}. \tag{3.8.48}$$

Substituting (3.8.47) to (3.8.46), we obtain

$$\begin{aligned}(v^2 z_1 - z_2) z_1 \frac{(v^2 z_2/z_1, v^{2n})_\infty}{(v^{2n} z_2/z_1, v^{2n})_\infty} \Xi_-(z_1) \Xi_- \left[B_{-k} + z_1^{-k}\right](z_2) \\ = (v^2 z_2 - z_1) z_2 \frac{(v^2 z_1/z_2, v^{2n})_\infty}{(v^{2n} z_1/z_2, v^{2n})_\infty} \Xi_-(z_2) \Xi_- \left[B_{-k} + z_2^{-k}\right](z_1)\end{aligned} \tag{3.8.49}$$

Consider expansion of LHS in z_2. Note that only non-negative degrees in z_2 appear. Analogously, expansion of RHS in z_1 has only non-negative degrees. Hence we can divide by $(v^2 z_1 - z_2)(v^2 z_2 - z_1)$ and obtain

$$\frac{(v^{2n+2} z_2/z_1, v^{2n})_\infty}{(v^{2n} z_2/z_1, v^{2n})_\infty} \Xi_-(z_1) \Xi_- \left[B_{-k} + z_1^{-k}\right](z_2) = \frac{(v^{2n+2} z_1/z_2, v^{2n})_\infty}{(v^{2n} z_1/z_2, v^{2n})_\infty} \Xi_-(z_2) \Xi_- \left[B_{-k} + z_2^{-k}\right](z_1) \tag{3.8.50}$$

Let us define

$$\tilde{\Xi}_-(z) = \Xi_-(z) \times \exp\left(\sum_{j>0} \frac{v^{2jn}}{j[n]^+_{v^j}} B_{-j} z^j\right) \tag{3.8.51}$$

We can substitute (3.8.51) to (3.8.50). Note that the exponent from (3.8.51) is an invertible series. Since (3.8.50) has only positive degree in both z_1 and z_2, we can multiply both sides by the inverse to the exponents. We obtain

$$\tilde{\Xi}_-(z_1) \tilde{\Xi}_- \left[B_{-k} + z_1^{-k}\right](z_2) = \tilde{\Xi}_-(z_2) \tilde{\Xi}_- \left[B_{-k} + z_2^{-k}\right](z_1) \tag{3.8.52}$$

It is legitimate to divide by $\tilde{\Xi}_-(z_1)\tilde{\Xi}_-(z_2)$. *A priori*, the result is a series in z_1 and z_2 with coefficients in rational function in B_{-j}. We obtain

$$\frac{\tilde{\Xi}_- \left[B_{-k} + z_1^{-k}\right](z_2)}{\tilde{\Xi}_-(z_2)} = \frac{\tilde{\Xi}_- \left[B_{-k} + z_2^{-k}\right](z_1)}{\tilde{\Xi}_-(z_1)} \tag{3.8.53}$$

In the RHS we have only positive powers of z_1 and negative powers of z_2, and vise versa for LHS. Hence, the expression is a constant. Therefore $\tilde{\Xi}(z)$ is a constant. Normalization condition (3.8.24) implies $\tilde{\Xi}(z) = 1$. $\square$

Conclusion

In this thesis, we have constructed explicit realizations of twisted Fock modules of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ and twisted W-algebras.

- In case $q_2 = 1$, we have constructed three realizations of twisted Fock module $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$: fermionic (Theorem 1.4.1), bosonic (Theorem 1.4.2) and strange bosonic (Theorem 1.4.3). It was proved that $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ acts via a quotient, isomorphic to twisted deformed W-algebra (Theorems 1.7.1 and 1.7.2). These results were generalized for representation obtained by *restriction to a sublattice* (Proposition 1.5.4 and Theorem 1.8.1). As an application, we have proved an identity for q-deformed conformal blocks (Theorem 1.9.3).

- We have constructed explicitly action of twisted and non-twisted Virasoro algebras on an integrable level 1 representation of quantum affine $\mathfrak{sl}_2$ (Theorems 2.4.1 and 2.4.2 correspondingly). The answer is expressed via vertex operators of quantum affine $\mathfrak{sl}_2$.

- We have constructed explicitly twisted Cherednik representation of double affine Hecke algebra $\mathcal{H}_N$ (Theorems 3.2.1 and 3.2.4). Twisted Fock module of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ is constructed explicitly via semi-infinite construction (Theorem 3.6.1). Action of Chevalley generators is expressed via vertex operators $U_v(\widehat{\mathfrak{gl}}_n)$. As a corollary, we have constructed an identification (as vector spaces) of representations F_i and $\mathcal{F}_u^\tau$.

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