17 July 2021, Skoltech, Center for Advanced Studies

Rules

- Test duration is 3 hours.
- There are 9 problems on 2 pages.
- Please, provide the key points of solutions for all problems. Correct answers are important, but not sufficient, we want to see that they are not consequences of the wrong solutions.
- You may use LaTeX notation to type your solutions: we can read it, or compile. Scans or photos of solutions are also fine, if the proctor allows it. In this case please send a separate file for each problem.

Problems

1. **Integral.**
   Compute Gaussian integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\} dx_1 dx_2$$

2. **Jordan blocks.**
   Denote by $J_n(\lambda)$ Jordan block of size $n \times n$ corresponding to eigenvalue λ. Find Jordan decomposition of the tensor product of two operators, $J_4(0)$ and $J_6(0)$:

$$J_4(0) \otimes J_6(0) = \bigoplus_i J_{k_i}(\lambda_i).$$

3. **Stokes theorem.**
   Write down a Lagrangian for the particle of mass m and electric charge e moving in the 3-dimensional magnetic field

$$\vec{B} = g\frac{\vec{r}}{|\vec{r}|^3}.$$

   Find a non-trivial integral of motion, different from energy.

4. **Generating function.**
   Let $a(n)$ be a number of partitions of n into sum of different positive odd integer summands. For example $a(0) = 1$ and $a(16) = 5$ since

$$16 = 1 + 15 = 3 + 13 = 5 + 11 = 7 + 9 = 1 + 3 + 5 + 7.$$

   Find generating function $\sum_{n=0}^{\infty} a(n)q^n$.

5. **Quantum mechanics.**
   Consider the Hamiltonian

$$\hat{H} = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right) + ax.$$

   Find its ground state energy up to corrections of order $O(a^5)$.

6. **Lie algebras.**
Lie algebra $\mathfrak{gl}_3$ of all 3×3 matrices contains Lie subalgebra $\mathfrak{so}_3$ of antisymmetric 3×3 matrices. Consider adjoint action of this $\mathfrak{so}_3$ on $\mathfrak{gl}_3$. Find the decomposition of $\mathfrak{gl}_3$ into the direct sum of irreducible representations of $\mathfrak{so}_3$ under such action. Find dimensions of corresponding irreducible representations.

7. **Fiber bundle.**
Let $E = \mathbb{CP}^3$ be a 3-dimensional complex projective space, and $B = S^4$ be a 4-dimensional sphere. Find any fiber bundle with total space E and base B. What is the fiber?

8. **Taylor series.**
Find radius of convergence of the Taylor series

$$\sum_{n=0}^{\infty} \frac{B_n z^n}{n!} = \frac{z}{e^z - 1}.$$

9. **Torus.**
Let T be a two-dimensional torus. The group G of order 3 acts on T in such a way that the quotient is homeomorphic to a sphere. Find the number of points in T invariant under the action of G.