Test for Master program in Mathematical and Theoretical Physics

April 2019

Note that only the answers are evaluated. Positive mark will be given only if complete and correct answer is given.

Test duration is 2 hours.

1. (a) Find the residue of the form $\frac{dz}{z}$ at 0.
   (b) Find the residue of the form $\frac{dz}{z}$ at ∞.
   (c) Find the residue of the form $\frac{dz}{z^3}$ at 0.
2. Find genus of the curve $w + \frac{1}{w} = p_n(x)$, where $p_n(x)$ is polynomial of degree n with generic coefficients.
3. Find non-trivial differential equation of minimal possible order with constant coefficients with solution $y(z) = z^5 e^{2z}$.
4. How many different (non-homeomorphic) closed surfaces can be obtained from the square by gluing all its edges in pairs (non necessarily pairs of opposite edges)? How many of them are nonorientable?
5. Compute the sum $\sum_{n=0}^{\infty} nq^n$, for $|q| < 1$
6. Let V be a standard three-dimensional irreducible representation of $SO(3)$. Find dimensions of irreducible submodules in $V^{\otimes 100}$. (Quantum mechanical particle of spin l corresponds to representation of dimension $2l+1$, tensor multiplication is summation of spins.)
7. Given system with Lagrangian $\mathcal{L}_1(\dot{\phi}, \phi) = \dot{\phi}^2/2 + \cos\phi$ (pendulum). Initial conditions at $t = 0$ are given by $\phi(0) = \pi - \epsilon$ and $\dot{\phi}(0) = 0$. Denote period of oscillation with such initial conditions by $T_1(\epsilon)$.
   (a) Find leading behavior (only the first relevant term) of the function $T_1(\epsilon)$ in the limit $\epsilon \to 0$, $\epsilon > 0$.
   (b) The same question for the system with Lagrangian $\mathcal{L}_2(\dot{\phi}, \phi) = \dot{\phi}^2/2 - \cos\phi$ and with the same initial conditions. Find leading behaviour of corresponding period of oscillations $T_2(\epsilon)$ in the limit $\epsilon \to 0$.
8. For the Hamiltonian $H(\hat{p}, \hat{q}) = (\hat{p} + il \tanh \hat{q})(\hat{p} - il \tanh \hat{q})$, where $l > 0$, $i^2 = -1$:
   (a) Find ground state wave function in coordinate representation $\psi_0(q)$ (eigenvector in $L^2(\mathbb{R})$ with the lowest eigenvalue) and ground-state energy E_0 (corresponding eigenvalue).
   (b) This Hamiltonian may be rewritten in a form $H(\hat{p}, \hat{q}) = \hat{p}^2 + U(\hat{q})$. What is $U(\hat{q})$?

   In coordinate representation $\hat{p} = -i\frac{d}{dq}$, $\hat{q} = q$.