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Contents

Introduction

Historical review

The system of one-dimensional particles with inverse-square pairwise interactions has played a great role in mathematical and theoretical physics for the past 40 years. This model arises and has different applications in various fields of physics, such as condensed matter physics, spin chains, gauge theory, and string theory and constitutes the main example of integrable and solvable many-body system. In the literature, it is labeled by the names of F. Calogero, B. Sutherland and Y. Moser. The system of identical particles scattering on the line with inverse-square potential was as first introduced by F. Calogero in 1971 [10]. Its Hamiltonian is

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i<j} \frac{g}{(q_i - q_j)^2}.$$

where we use the standard notations of momentums and coordinates. Here the particle masses are scaled to unity , g is the coupling constant. We consider a periodic version of the system (for example, with the period 2π), assuming that infinitely many images of particles interact, then the two-body potential becomes

$$V(x) = \sum_{n=-\infty}^{\infty} \frac{g}{(x+2\pi n)^2} = \frac{g}{2\sin\frac{x}{2}}.$$

This was introduced by B. Sutherland in 1971 [55]. It is convenient to use the following parametrization of the coupling constant:

$$g = \beta(\beta - 1).$$

We consider a system of N identical particles on a circle of length L, which we will call the quantum Calogero-Sutherland system, with the following Hamiltonian

$$H = -\sum_{i=1}^{N} \left(\frac{\partial}{\partial q_i}\right)^2 + 2\left(\frac{\pi}{L}\right)^2 \sum_{i<j}^{N} \frac{\beta(\beta-1)}{\sin^2\left(\frac{\pi}{L}(q_i - q_j)\right)}, \tag{0.1}$$

which is the main point of our research. It is natural to consider periodic wave functions of the system

$$\phi(q_1, \ldots, q_i + L, \ldots, q_N) = \phi(q_1, \ldots, q_i, \ldots, q_N).$$

The function

$$\phi_0(\mathbf{q}) = \phi_0(q_1, \ldots, \ldots, q_N) = \prod_{i<j} |\sin(\frac{\pi}{L}(q_i - q_j))|^\beta$$

represents the vacuum state with eigenenergy [23]

$$E_0 = (\pi\beta/L)^2\, N(N^2-1)/3.$$

Applying the transformation $\phi_0(\mathbf{q})^{-1} H \phi_0(\mathbf{q})$ and passing to the collective variables $x_i = e^{\frac{2\pi i q_i}{L}}$, we arrive to the effective Hamiltonian

$$H = \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i}\right)^2 + \beta \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\right). \tag{0.2}$$

The Hamiltonian (0.2) is a differential-difference operator. It turns out that there is a family of commuting differential-difference operators that includes (0.2). This family can be constructed using the Heckman-Dunkl operators [15, 17]. We give the expressions of them in the form suggested in [46]:

$$D_i^{(N)} = x_i \frac{\partial}{\partial x_i} + \beta \sum_{j \neq i} \frac{x_i}{x_i - x_j} (1 - K_{ij}), \tag{0.3}$$

where K_{ij} is a permutation operator. Symmetric polynomials in $D_i^{(N)}$ commute [17]. Denote by

$$H_k^{(N)} = \mathrm{Res}_+ \left(\sum_i \left(D_i^{(N)} \right)^k \right), \tag{0.4}$$

where Res_+ means a restriction on the space of symmetric polynomials. The operators $H_k^{(N)}$ can be chosen as the higher Hamiltonians of the Calogero-Sutherland model. In particular, $H = H_2^{(N)}$.

The eigenfunctions of commuting operators $H_k^{(N)}$ are symmetric polynomials in N variables with the parameter $\alpha = \frac{1}{\beta}$, which are called Jack polynomials [21]. They are parametrized by the partitions and constitute a generalization of Schur polynomials and a special case of symmetric Macdonald polynomials with two parameters q, t [30, 31]. Putting $q, t \to 1$ and assuming that $q = t^\alpha$, we obtain Jack polynomials. It is known a family of difference operators for which Macdonald polynomials are eigenfunctions [30]. In the case of Jack polynomials these operators were introduced by J. Sekiguchi [51] and A. Debiard [13]. The Sekiguchi-Debiard operators are degeneration of Macdonald operators. In fact, they do not coincide with the operators given in (0.4), but can be expressed as a polynomial in (0.4).

The construction of Macdonald polynomials and corresponding commuting difference operators is also known for an arbitrary root system [12, 32, 33]. A generalization of Jack polynomials for arbitrary root systems was introduced by G. Heckman and E. Opdam and is called Jacobi polynomials associated with the root system [18, 19, 20, 44]. Jack polynomials is associated with the root system A_n. We consider only this case. We remark that the Calogero- Sutherland system is an integrable system corresponding to the root system A_{N-1}, following M. Olshanetsky and A. Perelomov [42].

Naturally, there is a question about the description of the model where the number of particles N tends to infinity. In papers [4, 6, 7, 22, 46] from the 80's to early 90's there were presented the explicit answers for the limit of the second Hamiltonian (0.2) in the bosonic Fock space. About 20 years later, the general construction of commuting Hamiltonians in the bosonic Fock space was presented by M. Nazarov and E. Sklyanin [40] and independently by A. Veselov and A. Sergeev [52]. Developing Macdonald's ideas, M. Nazarov and E. Sklyanin in [40] found the expressions for Sekiguchi-Debyard operators in the limit where N tends to infinity. The main tool was the theory of symmetric functions. Symmetric functions can be considered as symmetric polynomials in infinite number of variables. The zero sector of the bosonic Fock space can be identified with the ring of symmetric functions, which is formally defined as the projective limit of rings of symmetric polynomials. Thus there was constructed a family of operators whose eigenfunctions are Jack symmetric functions.

In [39],[52] another construction of the limit for Calogero-Sutherland model in the bosonic Fock space was presented. The main idea was to use the family of Dunkl operators

(0.3) as a quantum L-operator of the system. For Calogero systems the L-operator was already known [37] and was similar to the action of the family of Dunkl operators, written in matrix form in a suitable basis. Thus a precise construction of higher Hamiltonians in the bosonic Fock space was suggested and this allowed to show that the limiting system is integrable. The resulting system can be considered as a quantum analogue of the integrable hierarchy of the Benjamin-Ono equation [1, 47].

For special value of the coupling constant the symmetric Jack functions become Schur functions , and the Benjamin-Ono equation respectively degenerates into the dispersionless KdV equation (or the so-called Burger's equation). The exact construction of commuting Hamiltonians of the quantum dispersionless KdV equation can be obtained directly from the boson-fermion correspondence and was presented by A. Pogrebkov in [45]. Hamiltonians can be obtained recurrently [45] or in terms of the generating function [41, 50].

We consider the spin Calogero-Sutherland systems which are generalizations of these models, where extra degrees of freedom are involved, which are usually interpreted as spin variables. Integrability of the Calogero system has been studied in numerous papers, see for example [29]. The Calogero-Sutherland spin system is superintegrable due to N. Reshetikhin [48, 49]. In this paper, we will use a special case of the spin model corresponding to the root system A_N and the representation of the higher weight of $\mathfrak{sl}_N$. In this case, the numerator of the potential of Hamiltonian (0.1) will be $\beta(\beta - K_{ij})$, where K_{ij} is the coordinate exchange operator of i-th and j-th particles, and the dependence on spin is implicit.

The spin CS system has the Yangian symmetry, in other words the Hamiltonians of the Calogero-Sutherland system commute with the Yangian action, moreover they are expressed through the central elements of the Yangian elements. The presence of Yangian symmetry is directly related to the Dunkl operators. They satisfy the relations of the degenerate affine Hecke algebra, which in turn allows us to construct the representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_s)$ according to the general construction [5, 14]. Thus, the higher Hamiltonians of the system can be chosen as the center of the Yangian, namely, as the coefficients of the quantum determinant.

In the symmetric case the limit expression N for the second Hamiltonian in collective variables was obtained in [7]. The antisymmetric limit of the spin system was studied by D. Uglov in [56, 58]. D. Uglov studied the projective properties of the Yangian action for a finite system, namely, he presented a formula of renornalization of the transfer matrix of the Yangian in order form a projective system and the action was stabilized. Also D. Uglov decomposed the corresponding Fock space into irreducible components with respect to the Yangian action and found the spectrum of Hamiltonians.

State of the problem and main results

The main purpose of this work is to study the limits of the Calogero-Sutherland system in the scalar and spin cases when the number of particles N tends to infinity. In each case we study the bosonic and fermionic limit corresponding to the symmetric and antisymmetric wave functions of the system. Here we list the results, further we give the precise formulations.

For the fermionic limit of the scalar system, we derive a limit expression for the Dunkl operator via free fermionic fields, see Theorem 2.1, which allows us to present the construction of commuting Hamiltonians in the Fock space, see Proposition 2.4. In the case of the value of the coupling constant $\beta = 0$, we get an explicit formula for the generating function of Hamiltonians that differs from the previously known ones. The first one is given as a bosonic normal ordered answer, see Proposition 3.1. The second formula is given in terms of simple integral operator, but is not normal ordered, see Proposition 3.2.

The spin CS system has the Yangian symmetries. In fact the action of Yangian generators as well as Hamiltonians in scalar case do not form a projective system. So we study the projective properties of the Yangian action and formulate the results in Proposition 4.1 and Proposition 4.2.

For spin system we realize the bosonic and fermionic limit in a multicomponent Fock space. We introduce the maps to finite system and construct the pullback of finite Dunkl operators in terms of vertex operators in bosonic case and in terms of free fermion fields in fermionic case, see Proposition 5.1 and Proposition 6.1. The limit of Dunkl operator allows to construct the corresponding Yangian representation in the Fock space, see Theorem 5.1 and Theorem 6.1. In the bosonic case we investigate the classical limit, see Propositions 5.3 and 5.4.

1. Bosonic limit of Calogero-Sutherland system. In the first section we review recent results [39, 52] concerning the bosonic limit of Calogero-Sutherland system and rewrite them in a language of vertex operators. We use the notations differing from [39, 52] but more convinient for our purpose and clearifying the further exposition.

We begin with the description of the finite CS system restricted on the ring of symmetric polynomials in N variables. The main idea is to regard the equivariant Heckman-Dunkl operators as a quantum L-operator acting on the space of polynomial functions of one variable with coefficients being symmetric polynomials of the remaining $N-1$ variables. Clearly, the Dunkl operator $D_i^{(N)}$ itself preserves the symmetry involving all variables other than x_i and therefore it acts on the space $\Lambda_+^{N,i}$ of functions symmetric in all variables except x_i.

The action of the higher Hamiltonian (0.4) can be obtained by the following procedure: we start with a symmetric polynomial $f(x_1,\dots,x_N) \in \Lambda_+^N$ and construct a vector of its N copies. The action the k-th power of Dunkl operator $(\tilde{D}_i^{(N)})^k$ provides a family of N equivariant functions: $f_i(x_1,\dots;x_i;\dots x_N) = (\tilde{D}_i^{(N)})^k f(x_1,\dots,x_N) \in \Lambda_+^{N,i}$ such that

- $f_i(x_1,\dots;x_i;\dots x_N)$ is a polynomial symmetric in all variables except x_i
- $$K_{ij} f_i = f_j. \tag{0.5}$$

For $g(x_1,\dots;x_i;\dots x_N) \in \Lambda_+^{N,i}$ we introduce an operator of its symmetrization

$$\mathrm{E}_N\, g = \sum_{j=1}^{N} K_{ij} g.$$

Then we apply E_N to a function f_i from an equivariant family (0.5):

$$\mathrm{E}_N\, f_i = \sum_{j=1}^{N} f_j$$

This procedure can be illustrated by the following matrix formula:

$$\tilde{H}_k = \left(1, 1, \ldots\right) \begin{pmatrix} x_1 \frac{\partial}{\partial x_1} + \beta \sum_{i=2}^{N} \frac{x_1}{x_1 - x_i} & -\beta \frac{x_1}{x_1 - x_2} & \ldots \\ -\beta \frac{x_2}{x_2 - x_1} & x_2 \frac{\partial}{\partial x_2} + \beta \sum_{i \neq 2} \frac{x_2}{x_2 - x_i} & \vdots \\ \vdots & \ldots & \ddots \end{pmatrix}^k \begin{pmatrix} f \\ f \\ \vdots \\ f \end{pmatrix},$$

which resembles the Lax matrix (see [37]) for CS system.

We reformulate the procedure in terms of the Newton polynomials $p_k^{(N)} = x_1^k + \cdots + x_N^k$ and express the Heckman-Dunkl operators via finite analogous $V_+(z)$, $V'_+(z)$ of the vertex operators $\Phi(z)$, $\Phi^{-1}(z)$ and the negative part of derivative of the bosonic field $\varphi^-(z)$, given by the formulas:

$$\Phi(z) = \exp\left(\sum_{n \geqslant 0} z^n \frac{\partial}{\partial p_n}\right), \qquad \varphi^-(z) = \left(\sum_{n \geq 0} \frac{p_n}{z^n}\right).$$

To do that we present a symmetric polynomial in the following form

$$f(p_1^{(N)}, p_2^{(N)}, p_3^{(N)}, \ldots)$$

The operator $V_+(x_i)$ changes each occurrence of a Newton sum $p_k^{(N)}$ by $p_k^{(N-1)} + x_i^k$, so $V_+(x_i) f \in \Lambda_+^{N,i}$ is a Taylor decomposition of polynomial f by variable x_i.

To symmetrize the function $F(x_i, \{p_n^{(N-1)}\}) \in \Lambda_+^{N,i}$ we use the the formal intagral

$$\mathrm{E}_N\, F(\{p_n\}) = \oint \frac{d\xi}{\xi} \varphi^-(\xi) \left(V'_+(\xi) F\right) (\xi; \{p_n\}).$$

which counts the residue at infinity. The operator $V'_+(\xi)$ changes each occurrence of a Newton sum $p_k^{(N-1)}$ by $p_k^{(N)} - \xi^k$. Then the integral $\oint \frac{d\xi}{\xi} \varphi^-(\xi)$ changes each item ξ^k by $p_k^{(N)}$.

In section 1.3 we realize the bosonic limit in the extended ring of symmetric functions $\hat{\Lambda}$. Let $\hat{\Lambda} = \Lambda[p_0]$ be a ring symmetric functions [30] extended by a free variable p_0.The space $\hat{\Lambda}$ is an irreducible representation of the Heisenberg algebra $\mathcal{H}$, generated by the elements p_n and $\dfrac{\partial}{\partial p_n}$ and can be regarded as a polynomial version of the Fock space. It contains the vacuum vector $|0\rangle_+$, such that

$$\frac{\partial}{\partial p_n} |0\rangle_+ = 0, \qquad n = 0, 1, \ldots.$$

The dual vacuum vector ${}_+\langle 0|$ satisfies the condition

$${}_+\langle 0| p_n = 0, \qquad n = 0, 1, \ldots.$$

We define a projection $\tilde{\pi}_N : \hat{\Lambda} \to \Lambda_+^N$ for an element $|v\rangle_+ \in \hat{\Lambda}$ as the following matrix element:

$$\tilde{\pi}_N |v\rangle_+ = {}_+\langle 0| \Phi(x_N) \ldots \Phi(x_2) \Phi(x_1) |v\rangle_+.$$

This projection maps p_k to the corresponding Newton polynomial in N variables:

$$\tilde{\pi}_N : p_k \to p_k^{(N)} = \sum_{i=1}^{N} x_i^k, \quad p_0 \to N.$$

We define a linear map $\mathcal{S} : \hat{\Lambda} \otimes \mathbb{C}[z] \to \hat{\Lambda}$ as

$$\mathcal{S}F(\{p_n\}) = \oint \frac{d\xi}{\xi} \varphi^-(\xi)\Phi^{-1}(\xi)F(\xi, \{p_n\}).$$

and prove that the map $\mathcal{S}$ is the pullback of the finite symmetrization E_N under the map $\tilde{\pi}_N$:

$$\mathrm{E}_N\, \tilde{\pi}_{N-1}F(z, \{p_n\}) = \tilde{\pi}_N \mathcal{S}(F(z, \{p_n\}).$$

We present the main result of this section:

Theorem 1.1 *The operator $\tilde{D} : \hat{\Lambda} \otimes \mathbb{C}[z] \to \hat{\Lambda} \otimes \mathbb{C}[z]$ given by*

$$\tilde{D}(F(z, \{p_n\})) = z\frac{\partial}{\partial z}F(z, \{p_n\}) + \beta z \oint \frac{d\xi}{\xi^2}\frac{1}{1 - \frac{z}{\xi}}\Phi^*(\xi)\Phi(z)F(\xi, \{p_n\}).$$

is a limit of Dunkl operators $\tilde{D}_i^{(N)}$.

In other words, the operator $\tilde{D}$ is a pullback of $\tilde{D}_i^{(N)}$ under the map $\tilde{\pi}_N$. This result was formulated before in [39, 52] in other terms, here we present the formula in the language of vertex operators. This theorem implies the following

Proposition 1.2 *The operators $\tilde{\mathscr{H}}_k = \mathcal{S}\tilde{D}^k\Phi(z) : \hat{\Lambda} \to \hat{\Lambda}$,*

$$\tilde{\mathscr{H}}_k : \hat{\Lambda} \xrightarrow{\Phi(z)} \hat{\Lambda} \otimes \mathbb{C}[z] \xrightarrow{D^k} \hat{\Lambda} \otimes \mathbb{C}[z] \xrightarrow{\mathcal{S}} \hat{\Lambda},$$

generate a commutative family of Hamiltonians of the limiting system [39, 52].

In section 1.4 we show that in classical limit this system becomes the Benjamin-Ono hierarchy following [39].

2. Fermionic limit of CS system.

The second section is devoted to the fermionic limit of Calogero-Sutherland model, we describe the results of paper [25]. In this case the particles are fermions and we deal with the antisymmetric wave functions.

As well as in bosonic case we begin with the description of the CS system restricted to the space of antisymmetric polynomials Λ_-^N in terms of Heckman–Dunkl operators. We then express Heckman–Dunkl operators via finite analogs $V_-(z)V_+(z)$ and $V'_-(z)V'_+(z)$ of vertex operators $\Psi(z)$ and $\Psi^*(z)$, where

$$\Psi(z) = z^{p_0} \exp\left(-\sum_{n>0} \frac{p_n}{nz^n}\right) \exp\left(\sum_{n\geq 0} z^n \frac{\partial}{\partial p_n}\right)$$

$$\Psi^*(z) = z^{-p_0} \exp\left(\sum_{n>0} \frac{p_n}{nz^n}\right) \exp\left(-\sum_{n\geq 0} z^n \frac{\partial}{\partial p_n}\right).$$

To do this we present any antisymmetric polynomial in N variables as

$$\prod_{i>j}(x_i - x_j) f(p_1^{(N)}, p_2^{(N)}, p_3^{(N)}, \dots)$$

where $p_k^{(N)} = x_1^k + \dots + x_N^k$. The operator $V_+(x_1)$ changes each occurrence of $p_k^{(N)}$ by $p_k^{(N-1)} + x_1^k$, while the operator

$$V_-(x_1) = x_1^N \exp\left(-\sum_{n>0} \frac{p_n^{(N-1)}}{n x_1^n}\right)$$

is the multiplication by $\prod_{i=2}^N (x_1 - x_i)$, so that the application of $V_-(x_1)V_+(x_1)$ to an antisymmetric polynomial $g(x_1, ..., x_N)$ is just its Taylor decomposition with respect to x_1. On the other hand, the operators $V'_-(z)V'_+(z)$ are used for the total antisymmetrization of the functions, antisymmetric with respect to all variables except one. This is done in Section 2.1.

In the next subsection we realize a limit in the polynomial Fock space $\hat{\Lambda}$. To each vector $|v\rangle$ of $\hat{\Lambda}$ we attach a family $\{\bar{\pi}_N(v)\}$ of antisymmentric functions of N variables, given by matrix elements

$$\bar{\pi}_N(v) = \langle 0|\Psi(x_N)\cdots\Psi(x_1)|v\rangle. \tag{0.6}$$

The goal is to construct operators in the space $\hat{\Lambda}$ which are compatible with finite CS Hamiltonians with respect to evaluation maps (0.6). This is done following E.Sklyanin ideology [39, 27]: we introduce an auxillary space $U \subset \mathbb{C}[z, z^{-1}]] \otimes \hat{\Lambda}$ and its evaluations to the spaces of polynomials antisymmetric with respect to all variables except one. We present operators, acting in U which are compatible with the above evaluation maps.

The key point of the construction is an operator of integral average $\mathcal{A} : \mathbb{C}[z, z^{-1}]] \otimes \hat{\Lambda} \to \hat{\Lambda}$, which is the limiting analogue of finite antisymmetrization. Let $F(z) \in \mathbb{C}[z, z^{-1}]] \otimes \hat{\Lambda}$, then we define $\mathcal{A}F \in \hat{\Lambda}$ by the following formula:

$$\mathcal{A}F = \frac{1}{(2\pi i)^2} \int_{z\circlearrowleft 0} dz \int_{u\circlearrowleft z} du \frac{\Psi^*(u)F(z)}{u - z}.$$

In Lemma 2.3 we show that $\mathcal{A} : U \to \hat{\Lambda}$ is a pullback of finite antisymmetrization.

Further we define an operator $D : \mathbb{C}[z, z^{-1}]] \otimes \hat{\Lambda} \to \mathbb{C}[z, z^{-1}]] \otimes \hat{\Lambda}$ by the following formula

$$DF(z) = z\frac{\partial}{\partial z}F(z) + \beta \frac{1}{(2\pi i)^2} \int_{w\circlearrowleft 0} dw \int_{u\circlearrowleft w} \frac{du}{(u-w)} \frac{\Psi^*(u)}{\left(1 - \frac{w}{z}\right)} \left(\Psi(w)F(z) - \Psi(z)F(w)\right).$$

and prove

Theorem 2.1 *The operator D acting on the auxillary space U is a pullback of Heckman–Dunkl operators $D_i^{(N)}$ under the map $\bar{\pi}_N$.*

The Hamiltonians of finite system with N particles in antisymmetric case can be expressed by meas of Dunkl operators analogously (0.4):

$$\bar{H}_k^{(N)} = \mathrm{Res}_-\left(\sum_i \left(D_i^{(N)}\right)^k\right),$$

where Res means the restriction on the space of antysymmetric polynomials Λ_-^N. We construct the limiting Hamiltonians $\mathscr{H}_k$ which are the pullbacks of finite Hamiltonians $\bar{H}_k^{(N)}$:

$$\bar{H}_k^{(N)}\bar{\pi} = \bar{\pi}\mathscr{H}_k.$$

We define the operators

$$\mathscr{H}_k = \mathcal{A}D^k\Psi(z) : \hat{\Lambda} \to \hat{\Lambda} \tag{0.7}$$

and formulate

Proposition 2.4 *The operators $\mathscr{H}_k$ generate a commutative family of operators in the space $\hat{\Lambda}$.*

The constructed Hamiltonians form a commutative family of operators in the space $\hat{\Lambda}$. Moreover, they commute inside the Heisenberg algebra and thus can be used as well in its other representations, for instance, in the bosonic Fock space. We can define the projection $\pi_N : \mathcal{F} \to \Lambda_-^N$ similar to (0.6)

$$\pi_N(v) = \langle 0|\Psi(x_N)\cdots\Psi(x_1)|v\rangle.$$

In fact it is nonzero only on the N-th sector $\mathcal{F}_N$ of the Fock space. Now the constructed Hamiltonians $\mathscr{H}_k$ are compatible with respect to the maps π_N, the commutativity $\pi_N\mathscr{H}_k = \bar{H}_k^{(N)}\pi_N$ is nontrivial on the N-th sector $\mathcal{F}_N$. We reformulate the same construction in the fermionic Fock space represented as space of semi-infinite wedges, we define the projection analogous to π_N which acts as a "cutting" of the wedge. We discuss this in Section 2.3.

3. Generating functions of commuting Hamiltonians for some special values of coupling constant.

In this section we consider the special case $\beta = 0$ of antisymmetric limit. (we use the notations for Hamiltonians as in previous section, where we assume $\beta = 0$). In this case the Hamiltonians (0.7) can be simply expressed as operators on the fermionic Fock space

$$\mathscr{H}_n = \sum_k \vdots k^n\psi_k^*\psi_k\vdots.$$

The boson-fermion correspondence allows to express $\mathscr{H}_n$ in the bosonic Fock space, it was done by A. Pogrebkov [45] for the additive version and later by P. Rossi [50] on the circle.

Here we derive the two formulas for the densities for $\mathscr{H}_n$ that was not known before, we present the results given in [35]. In case $\beta = 0$ the Dunkl operator is simply the differential operator $\left(z\frac{\partial}{\partial z}\right)$ and the Hamiltonians (0.7) are expressed from the densities $\mathscr{H}_k = \frac{1}{2\pi i}\int_{z\circlearrowleft 0}\frac{dz}{z}\mathscr{W}_k(z)$ which is given by

$$\mathscr{W}_k(z) = \frac{1}{2\pi i}\int_{u\circlearrowleft z} du\frac{\Psi^*(u)}{u-z}\left(z\frac{\partial}{\partial z}\right)^k\Psi(z).$$

These Hamiltonians are the pullbacks of simple differential operators $\bar{H}_k^{(N)} = \sum_{i=1}^N\left(x_i\frac{\partial}{\partial x_i}\right)^k$. We derive first formula by calculating the integral in variable u in $\mathscr{W}_k(z)$ using the bosonic calculus. This gives the following answer

Proposition 3.1 *The exponential generating function $\mathscr{W}(z,x)$ for densities $\mathscr{W}_k(z)$ is given by the formula*

$$\mathscr{W}(z,x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\mathscr{W}_k(z) = \frac{:\exp\left(x\left(z\frac{\partial}{\partial z} + \varphi(z)\right)\right): -1}{e^x - 1}$$

and satisfies the differential equation

$$\frac{\partial \mathscr{W}(x,z)}{\partial x} =: \varphi(z)\mathscr{W}(z,x): + z\frac{\partial \mathscr{W}(x,z)}{\partial z} - \frac{e^x \mathscr{W}(z,x) - \varphi(z)}{e^x - 1}.$$

Here the exponent of operator means the formal series acting on the identity:

$$\exp\left(x\left(z\frac{\partial}{\partial z} + \varphi(z)\right)\right) = 1 + x\varphi(z) + \frac{x}{2!}\left(\varphi^2(z) + z\frac{\partial}{\partial z}\varphi(z)\right) + \ldots$$

The second formula can be obtained by fermionic calculus and expressed in terms of integral operator. We introduce an integral operator $K : \mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]] \to \mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$, given by the formula

$$K\left[f(z)\right] = \frac{1}{2\pi i}\int\limits_{w\circlearrowleft z} \frac{dw}{w-z}\varphi(w)f(z),$$

here $f(z) \in \mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$. Then we present the explicit formulas for the Hamiltonians by the following

Proposition 3.2 *The exponential generating function for the Hamiltonians is given by*

$$\mathscr{H}(x) = \frac{1}{2\pi i(e^x - 1)}\int\limits_{z\circlearrowleft 0} dz \frac{e^{xK} - 1}{K}\left[\frac{\varphi(z)}{z}\right].$$

The Hamiltonians can be expressed by the formula:

$$\mathscr{H}_n = \frac{1}{2\pi i}\int\limits_{z\circlearrowleft 0} dz\left(\frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l+1}B_{n-l}K^l\left[\frac{\varphi(z)}{z}\right]\right).$$

Here B_n mean the Bernoulli numbers and the operator $\frac{e^{xK}-1}{K}$ means a formal power series in K:

$$\frac{e^{xK}-1}{K} = x + \frac{x^2}{2}K + \frac{x^3}{6}K^2 + \frac{x^4}{24}K^3 + \ldots.$$

We note that the answer for the Hamiltonians given in Proposition 3.2 is not normal ordered.

The Hamiltonians $\mathscr{H}_n$ commute, thus we can derive an hierarchy of time evolutions defined by these commutative flows as

$$\varphi_{t_n}(z) = [\mathscr{H}_n, \varphi(z)].$$

We derive the explicit formulas and formulate the result by the following

Lemma 3.5 *The hierarchy of time evolutions defined by commutative family (3.2) is given by*

$$\varphi_{t_k}(z) = \frac{1}{2} B(x) : \int_{x \circlearrowleft 0} dx \frac{k!}{x^{k+1}} \sinh\left(xz\frac{\partial}{\partial z}\right) e^{xS(xz\frac{\partial}{\partial z})\varphi(z)} : .$$

The classical limit of this hierarchy is the dispersionless KdV hierarchy on the circle [45].

4. Dunkl operators and representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_s)$**.**
The phase space of the quantum spin Calogero-Sutherland (CS) system consists of functions with values in vector space $(\mathbb{C}^s)^{\otimes N}$ while the dependence on spin in the Hamiltonian

$$H^{CS} = -\sum_{i=1}^{N} \left(\frac{\partial}{\partial q_i}\right)^2 + \sum_{i,j=1}^{N} \frac{\beta(\beta - K_{ij})}{\sin^2(q_i - q_j)},$$

is implicit [23]. Here K_{ij} is the coordinate exchange operator of particles i and j. After conjugating by the function $\prod_{i<j} |\sin(q_i - q_j)|^\beta$ which represents the degenerated vacuum state, and passing to the exponential variables $x_i = e^{2\pi i q_i}$ and the parameter $\alpha = \beta^{-1}$ more common in mathematical literature, we arrive after simple rescaling to the effective Hamiltonian

$$H = \alpha \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i}\right)^2 + \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\right) - 2 \sum_{i<j} \frac{x_i x_j}{(x_i - x_j)^2} (1 - K_{ij}),$$

which we restrict to the spaces $\Lambda_{\pm}^{s,N}$ of total invariants or respectively skewinvariants of the symmetric group S_N in the space $V^{\otimes N}$,

$$\Lambda_{\pm}^{s,N} = \left(V^{\otimes N}\right)^{(\pm)}.$$

Here $V = \mathbb{C}[z] \otimes \mathbb{C}^s$. The (skew)invariants are taken with respect to the diagonal action of the symmetric groups, $\sigma_{ij} \mapsto K_{ij} P_{ij}$, where K_{ij} is as above and P_{ij} is the permutation of i-th and j-th tensor copy of the vector space $\mathbb{C}^s$.

Further we use the Heckman–Dunkl operators $\mathcal{D}_i^{(N)} : V \otimes \Lambda_{\pm}^{s,N-1} \to V \otimes \Lambda_{\pm}^{s,N-1}$ in the form suggested by Polychronakos [46]:

$$\mathcal{D}_i^{(N)} = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{j \neq i} \frac{x_i}{x_i - x_j} (1 - K_{ij}).$$

These operators satisfy the relations

$$K_{ij} \mathcal{D}_i^{(N)} = \mathcal{D}_j^{(N)} K_{ij},$$
$$[\mathcal{D}_i^{(N)}, \mathcal{D}_j^{(N)}] = (\mathcal{D}_j^{(N)} - \mathcal{D}_i^{(N)}) K_{ij},$$

which coincide with the relations of the degenerate affine Hecke algebra H_N. By Drinfeld duality [14], this representation of degenerate affine Hecke algebra transforms to the representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_s)$ in $\Lambda_{\pm}^{s,N}$, see [5, 28]

$$t_{ab}(u) = \delta_{ab} + \sum_i \frac{E_{ab,i}}{u \pm \mathcal{D}_i^{(N)}}. \tag{0.8}$$

Here $E_{ab,i}$ describes the action of $\mathfrak{gl}_s$ on i-th tensor component,

$$E_{ab,i}\Big(\ldots\otimes\underbrace{(e^c\otimes x^k)}_{i}\otimes\ldots\Big)=\delta_{bc}\Big(\ldots\otimes\underbrace{(e^a\otimes x^k)}_{i}\otimes\ldots\Big).$$

and $t_{ab}(u)$, $a,b=1,\ldots s$,

$$t_{ab}(u)=\delta^{ab}+\sum_{i=0}^{\infty}t_{ab,i}u^{-i-1}$$

are generating functions of the generators $t_{ab,i}$ of the Yangian $\mathrm{Y}(\mathfrak{gl}_s)$. The defining relations of $\mathrm{Y}(\mathfrak{gl}_s)$ are [36]

$$[t_{ab}(u),t_{cd}(v)]=\frac{t_{cb}(u)t_{ad}(v)-t_{cb}(v)t_{ad}(u)}{u-v}.$$

Then the higher Hamiltonians of spin CS system can be chosen as coefficients of the quantum determinant

$$q\det t(u)=\sum_{\sigma\in S_m}(-1)^{sgn(\sigma)}t_{\sigma(1),1}(u)t_{\sigma(2),2}(u-1)...t_{\sigma(m),m}(u-m+1).$$

which generate the center of the $\mathrm{Y}(\mathfrak{gl}_s)$ [9, 36].

Our main goal is to construct the limit of the above Yangian action when N tends to infinity. In particular, we get the limits of the above commuting family of Hamiltonians. To construct the limit we need investigate the projective properties of the Yangian actions in phase spaces $\Lambda_{\pm}^{s,N}$ of CS model. Such an analysis was done by D.Uglov in [57], but our description differs from that of [57].

The rings Λ_+^N of scalar symmetric functions form the projective system with respect to the maps

$$\omega_N^+:\Lambda_+^N\to\Lambda_+^{N-1},\qquad \omega_N^+f(x_1,\ldots,x_N)=f(x_1,\ldots,x_{N-1},0).$$

Analogously, the spaces Λ_-^N of scalar skewsymmetric functions form the projective system with respect to the maps

$$\omega_N^-:\Lambda_-^N\to\Lambda_-^{N-1},\qquad \omega_N^-f(x_1,\ldots,x_N)=(x_1\ldots x_{N-1})^{-1}f(x_1,\ldots,x_{N-1},0).$$

Contrary to the ring of symmetric functions, the space $\hat{\Lambda}$ is not the projective limit of the spaces of (skew)symmetric functions due to the presence of zero mode p_0. On the other hand, CS Hamiltonians H_k themselves in both symmetric and skewsymmetric cases do not form a projective family since they do not respect natural projections

$$\omega_N^+H_k^{(N+1)}\neq H_k^{(N)}\omega_N^+.$$

Let

$$T(u)=\sum_{a,b=1}^{s}E_{ab}\otimes t_{ab}(u)\in\mathrm{End}(\mathbb{C}^s)\otimes\mathrm{Y}(\mathfrak{gl}_s)[u^{-1}]$$

be the generating matrix of Yangian generators. Denote by $T_N(u)$ the transfer matrix corresponding to the representation (0.8) , here index N denotes the number of particles. In scalar case ($s=1$) the transfer matrix $T_N(u)$ is the generating function of the Hamiltonians

$$T_N(u)=1+\frac{1}{u}H_0^{(N)}+\frac{1}{u^2}H_1^{(N)}+\frac{1}{u^3}H_2^{(N)}+\ldots$$

We formulate the projective property of $T_N(u)$ in scalar symmetric and skewsymmetric case

Proposition 4.1 *(i) In scalar symmetric case we have the following identity of operators from $\tilde\Lambda_+^N[u^{-1}] \to \tilde\Lambda_+^{N-1}[u^{-1}]$:*

$$\omega_N^+ T_N(u) = \frac{u+1}{u} T_{N-1}(u+1)\omega_N^+;$$

(ii) In scalar skewsymmetric case the following identity of operators from $\tilde\Lambda_-^N[u^{-1}] \to \tilde\Lambda_-^{N-1}[u^{-1}]$ holds:

$$\omega_N^- T_N(u) = \frac{u+1}{u} T_{N-1}(u-\alpha-1)\omega_N^-.$$

Iterating the relations from Proposition 4.1 we see, that the renormalized transfer matrices $\tilde T_N(u)$ and $\bar T_N(u)$ in symmetric and skewsymmetric case

$$\tilde T_N(u) = \frac{u-N}{u} T_N(u-N) \qquad \bar T_N(u) = T_N(u+\gamma N)\prod_{k=1}^{N}\frac{u+k\gamma}{u+k\gamma+1},$$

are compatible with projection maps ω_N^+ and ω_N^-, respectively

$$\omega_N^+\tilde T_N(u) = \tilde T_{N-1}(u)\omega_N^+ \qquad \omega_N^-\bar T_N(u) = \bar T_{N-1}(u)\omega_N^-$$

Here $\gamma = \alpha+1$. The coefficients of renormalized transfer matrices can be chosen as a projective system of Hamiltonians of CS system.

The statement of Proposition 4.1 can be generalized to skewsymmetric spin case. Regard an element f of $\Lambda_-^{s,N}$ as $(\mathbb{C}^s)^{\otimes N}$ valued function $f = f(x_1, x_2, \dots, x_N)$. We define a linear map $\omega_N : \Lambda_-^{s,N} \to \Lambda_-^{s,N-s}$ by the formula

$$\omega_N^-(f) = (x_1\cdots x_{N-s})^{-1}\left(1^{\otimes(N-s)}\otimes e_1^\perp\otimes e_2^\perp\cdots\otimes e_s^\perp\right) f(x_1,\dots,x_{N-s},0,\dots 0)$$

and formulate the following

Proposition 4.2 *The following identities of operators from $\mathbb{C}^s\otimes\tilde\Lambda_-^{s,Ns}[u^{-1}] \to \mathbb{C}^s\otimes \tilde\Lambda_-^{s,(N-1)s}[u^{-1}]$ holds:*

$$\omega_{Ns}^- T_{Ns}(u) = \frac{u+1}{u} T_{(N-1)s}(u-\alpha-s)\omega_{Ns}^-.$$

Set $\gamma = \alpha+s$ and

$$\bar T_{Ns}(u) = T_{Ns}(u+\gamma N)\prod_{k=1}^{N}\frac{u+k\gamma}{u+k\gamma+1},$$

treated as asymptotical series in u^{-1}. Then $\bar T_{Ns}(u)$ satisfy compatibility conditions

$$\omega_{Ns}^-\bar T_{Ns}(u) = \bar T_{(N-1)s}\omega_{Ns}^-$$

and form a projective system of transfer matrices.

5. Bosonic limit of spin Calogero-Sutherland system.

In this section we observe the results of [27] using slightly different language.

Let $\mathcal{H}^s$ be the Heisenberg algebra with generators $a_{c,k}, c = 1, \dots, s,\ k = 0, 1, ...$ and $(q_c)^{\pm 1}$, which satisfy the relations

$$[a_{c,k}, a_{d,l}] = k\delta_{cd}\delta_{k,-l}, \qquad q_c a_{d,k} = (a_{d,k} + \delta_{cd}\delta_{k0})q_c.$$

Let $\hat{\Lambda}^{(s)}$ be a representation of the Heisenberg algebra $\mathcal{H}^s$ with the vacuum vector $|0\rangle_+$ such that

$$a_{c,k}|0\rangle_+ = 0,\ c = 1, ..., s, \quad k > 0, \qquad q_c|0\rangle_+ = |0\rangle_+,\ c = 1, ..., s.$$

Denote by ${}_+\langle 0|$ the vector of the dual space, which satisfies the relations

$${}_+\langle 0|a_{c,k} = 0, \qquad c = 1, ..., s, \quad k \leq 0.$$

For any $|v\rangle_+ \in \hat{\Lambda}^{(s)}$ consider the matrix element $\tilde{\pi}_N(|v\rangle_+) \in V^{\otimes N}$

$$\tilde{\pi}_N(|v\rangle_+) = {}_+\langle 0|\mathbf{\Phi}(z_N)\mathbf{\Phi}(z_{N-1}) \cdots \mathbf{\Phi}(z_1)|v\rangle_+,$$

where

$$\begin{aligned} \Phi_c(z) &= \exp\left(\sum_{n>0} \frac{a_{c,n}}{n} z^n\right) q_c : \qquad && \hat{\Lambda}^{(s)} \to \hat{\Lambda}^{(s)} \otimes \mathbb{C}[z], \qquad c = 1, ..., s, \\ \mathbf{\Phi}(z) &= \sum_c \Phi_c(z) \otimes e_c : && \hat{\Lambda}^{(s)} \to \hat{\Lambda}^{(s)} \otimes V, \end{aligned}$$

are the vertex operators and by $\mathbf{\Phi}(z_k)$ we shortly denote $\mathbf{\Phi}(z_k) \otimes 1^{\otimes k-1}$. We show that $\tilde{\pi}_N(|v\rangle_+) \in \Lambda_+^{s,N}$ is symmetric invariant.

Our goal is to pull back the Yangian action (0.8) in $\Lambda_+^{s,N}$ through the map $\tilde{\pi}_N$. We use the similar procedure as in scalar case and decompose the application of each Yangian generator (0.8) to a vector $|w\rangle_+ \in \Lambda_+^{s,N}$ into several steps. First we present the symmetric tensor $|w\rangle_+ \in \Lambda_+^{s,N}$ as an element of $(\mathbb{C}[x_i] \otimes \mathbb{C}^s) \otimes \Lambda_+^{s,N-1}$ for each tensor component, producing an equivariant family of vectors, then we apply the power of Heckman operator $\mathcal{D}_i^{(N)}$ to the i-th vector of this equivariant family and get another equivariant family. The last step is the symmetrization $E_N(u)$ — the sum of all members of the equivariant family:

$$E_N(u) = \sum_{j=1}^N \sigma_{1j}(u),$$

where $\sigma_{ij} = K_{ij}P_{ij}$ is the permutation of i-th and j-th tensor factors.

For each $\boldsymbol{F}(z) \in \hat{\Lambda}^{(s)} \otimes V$ define the element $\mathcal{S}(\boldsymbol{F}(z)) \in \hat{\Lambda}^{(s)}$ as the formal integral

$$\mathcal{S}(\boldsymbol{F}(z)) = \frac{1}{2\pi i} \oint \frac{dz}{z} \mathbf{\Phi}^*(z)\boldsymbol{F}(z),$$

which counts zero term of the Laurent series. Here

$$\mathbf{\Phi}^*(z) = \sum_c \varphi_c^-(z) \cdot \Phi_c^{-1}(z) \otimes e_c^\perp : \qquad \hat{\Lambda}^{(s)} \otimes V \to \hat{\Lambda}^{(s)} \otimes \mathbb{C}[z],$$

the series $\varphi_c^-(z) = \sum_{n\le 0} a_{c,n} z^n$ and the operator $e_c^\perp : \mathbb{C}^s \to \mathbb{C}$ is given by the relation $e_c^\perp(e_b) = \delta_{bc}$. The key point of the construction is the following lemma which establishes the map $\mathcal{S}$ as the pullback of the finite symmetrization:

Lemma 5.2 *For each $\boldsymbol{F}(z) \in \hat{\Lambda}^{(s)} \otimes V$ and any natural N we have the equality of elements of $\Lambda_+^{s,N}$:*

$$E_N(\tilde{\pi}_{N-1} \otimes 1)(\boldsymbol{F}(z)) = \tilde{\pi}_N \mathcal{S}(\boldsymbol{F}(z)).$$

Let $\tilde{\mathcal{D}} : \hat{\Lambda}^{(s)} \otimes V \to \hat{\Lambda}^{(s)} \otimes V$ be the linear map, such that for any $\boldsymbol{F}(z) \in \hat{\Lambda}^{(s)} \otimes V$

$$\tilde{\mathcal{D}} \boldsymbol{F}^{(1)}(z) = \alpha z \frac{d}{dz} \boldsymbol{F}^{(1)}(z) + \frac{z}{2\pi i} \oint \frac{d\xi}{\xi^2 (1 - \frac{z}{\xi})} \boldsymbol{\Phi}^{*(2)}(\xi) \boldsymbol{\Phi}^{(2)}(z) \boldsymbol{F}^{(1)}(\xi)$$

Here the upper index (i), $i = 1, 2$ indicates in which tensor copy of $\mathbb{C}^s$ the corresponding vector lives or an operator acts. We state that the operator $\tilde{\mathcal{D}}$ is the pullback of the equivariant family of Heckman operators $\mathcal{D}_i^{(N)}$.

Proposition 5.1 *For any $\boldsymbol{F}(z) \in \hat{\Lambda}^{(s)} \otimes V$ we have*

$$(\tilde{\pi}_{N-1} \otimes 1)\, \tilde{\mathcal{D}}(\boldsymbol{F}(x_1)) = \mathcal{D}_1^{(N)}\, (\tilde{\pi}_{N-1} \otimes 1)\, \boldsymbol{F}(x_1)$$

Let $E_{ab} \in \operatorname{End} \mathbb{C}^s$, be the matrix unit, $E_{ab}(e_c) = \delta_{bc} e_a$. Denote by $\mathcal{E}_{ab}$, the operator $1 \otimes 1 \otimes E_{ab} : \hat{\Lambda}^{(s)} \otimes V \to \hat{\Lambda}^{(s)} \otimes V$:

$$\mathcal{E}_{ab} \boldsymbol{F}(z) = F_b(z) \otimes e_a.$$

Summarazing the statements above we get the following result [27]

Theorem 5.1 *The operator $T_{ab,n}$ given by*

$$T_{ab,n} = \frac{(-1)^n}{2\pi i} \oint \frac{dz}{z} \boldsymbol{\Phi}^*(z) \mathcal{E}_{ab} \tilde{\mathcal{D}}^n \boldsymbol{\Phi}(z)$$

is the pullback of the Yangian generator $t_{ab,n}$, see (0.8):

$$\tilde{\pi}_N T_{ab,n} = t_{ab,n} \tilde{\pi}_N \qquad \textit{for any} \qquad N \in \mathbb{N}.$$

Using this construction we derive the explicit expressions for the first Hamiltonians of CS system.

In the next section we investigate the classical limit of the system. We introduce the operator $\mathscr{H}^{\mathrm{cl}}$ which is the classical limit of the second Hamiltonian, the rule between the quantum commutator and Poisson bracket is $\beta^{-1}[\ ,\ ] \to \{\ ,\ \}$. In Proposition 5.3 we present the equations of motion determined by $\mathscr{H}^{\mathrm{cl}}$:

$$\frac{d\phi_a(z)}{dt} = \{\phi_a(z), \mathscr{H}^{\mathrm{cl}}\}.$$

Here and further $\phi_a(z)$ and $\mathcal{V}_a(z)$ are the classical analogues of field $\varphi_a(z)$ and vertex operator $\Phi_a(z)$ respectively.

The quantum system is integrable: it has an infinite number of integrals of motion that can be obtained from the q-determinant of the Yangian generator function $T_{ab}(u)$. It is natural to assume that the classical system is integrable as well. In particular, it should admit a Lax pair presentation. Consider the operators L and M acting on the analytic function $f(z)$:

$$Lf = z\frac{\partial}{\partial z}f(z) + \sum_a \mathcal{V}_a(z)\left(\phi_a^-(z)\mathcal{V}_a^{-1}(z)f(z)\right)^+,$$

$$Mf = \left(z\frac{\partial}{\partial z}\right)^2 f(z) + 2\sum_b \left(\phi_b^+(z)\phi_b^-(z)\right)^+ f(z) + 2\sum_b \mathcal{V}_b(z)z\frac{\partial}{\partial z}\left(\phi_b^-(z)\mathcal{V}_b^{-1}(z)f(z)\right)^+.$$

Proposition 5.4 *The operators L and M represent a Lax pair of the classical system:*

$$\frac{dL}{dt} = [M, L].$$

6. Fermionic limit for spin system.

The fermionic limit of spin CS system was studied by D. Uglov. Here we suggest and develop another approach, which leads to the limiting integrable system closely related to [57], but realized by free fermionic fields, we mainly follow [26].

We start from the fermionic Fock space $\mathcal{F}^s$, which is the representation of algebra $\mathcal{H}_-^s$ of s free fermion fields. We denote by $\Psi_c(z)$ and $\Psi_c^*(z)$ be the following generating functions of elements of $\mathcal{H}_-^s$:

$$\Psi_c(z) = \sum_{n\in\mathbb{Z}} \psi_{cn} z^n, \qquad \Psi_c^*(z) = \sum_{n\in\mathbb{Z}} \psi_{cn}^* z^{n-1}.$$

For any $|v\rangle \in \mathcal{F}^s$ we define a matrix coefficients by the following formula

$$\pi_N\left(|v\rangle\right) = \langle 0|\boldsymbol{\Psi}(z_N)\boldsymbol{\Psi}(z_2)\cdots\boldsymbol{\Psi}(z_1)|v\rangle, \qquad |v\rangle \in \mathcal{F}^s$$

where $\boldsymbol{\Psi}(z) = \sum_{c=1}^s \Psi_c(z)\otimes e_c$ and $e_c \in \mathbb{C}^s$ are again basic vectors of $\mathbb{C}^s$. The matrix element $\pi_N\left(|v\rangle\right)$ belongs to the space $\Lambda_-^{s,N}$, which is the phase space of finite spin CS system. Then we systematically construct the pullback with respect to the maps π_N of all operation required for the construction of the Yangian action on the finite-dimensional spin CS system.

The crucial point of the construction is the operator which the pullback of total finite antisymetrization $\mathrm{A}_N : V\otimes\Lambda_-^{s,N-1} \to \Lambda_-^{s,N}$, given by

$$\mathrm{A}_N(u) = u - \sum_{j=2}^N \sigma_{1j}(u).$$

For each $\boldsymbol{F}(z) \in \mathcal{H}_-^s(z)\otimes\mathbb{C}^s$ define the element $\mathcal{A}(\boldsymbol{F}(z)) \in \mathcal{H}_-^s$ as the integral

$$\mathcal{A}(\boldsymbol{F}(z)) = \frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0} dz \int_{u\circlearrowleft z} du \frac{\boldsymbol{\Psi}^*(u)\boldsymbol{F}(z)}{u-z}.$$

The following lemma establishes the map $\mathcal{A}$ as the pullback of the finite antisymmetrization:

Lemma 6.2 *For each $\boldsymbol{F}(z) \in \mathcal{H}^s_-(z) \otimes \mathbb{C}^s$ satisfying the conditions (6.25) and (6.26), any $|v\rangle \in \mathcal{F}^s$ and any natural N we have the equality of elements of $\Lambda^{s,N}_-$:*

$$\mathrm{A}_N\, \pi_{N-1,1}(\boldsymbol{F}(z) \otimes |v\rangle) = \pi_N \mathcal{A}(\boldsymbol{F}(z))|v\rangle.$$

We remark that Lemma 6.2 holds only for a subspace of $\mathcal{H}^s_-(z) \otimes \mathbb{C}^s$ analogously with the fermionic scalar case. The special conditions (6.25) and (6.26) are preservation of the polynomial space and homogeneity. These conditions are preserved by pullbacks of Dunkl operators, which we define further.

Define an operator $\mathcal{D} : \mathcal{H}^s_-(z) \otimes \mathbb{C}^s \to \mathcal{H}^s_-(z) \otimes \mathbb{C}^s$ by the relation

$$\mathcal{D}\boldsymbol{F}(z) = \alpha z \frac{d}{dz}\boldsymbol{F}(z) +$$
$$\frac{z}{(2\pi i)^2} \int_{\substack{w \circlearrowleft 0 \\ |w|<|z|}} dw \int_{u \circlearrowleft w} du\ \boldsymbol{\Psi}^{*(2)}(u) \frac{\boldsymbol{\Psi}^{(2)}(w)\boldsymbol{F}^{(1)}(z) - \boldsymbol{\Psi}^{(2)}(z)\boldsymbol{F}^{(1)}(w)}{(u-w)(z-w)}.$$

By means of Lemma 6.2 we now can identify the operator $\mathcal{D}$ as a pullback of the equivariant family of Heckman operators $\mathcal{D}^{(N)}_i$ acting in the space of partially antisymmetric tensors

Proposition 6.1 *For any $\boldsymbol{F}(z) \in \mathcal{H}^s_-(z) \otimes \mathbb{C}^s$ satisfying the condition (6.25) and (6.26), $|v\rangle \in \mathcal{F}^s$ and $N \in \mathbb{N}$ we have the equality*

$$\pi_{N-1,1}(\mathcal{D}\boldsymbol{F}(x_1) \otimes |v\rangle) = \mathcal{D}^{(N)}_1 \pi_{N-1,1}(F(x_1) \otimes |v\rangle).$$

As in bosonic spin case we introduce operators :

$$\mathrm{T}_{ab,n} = \mathcal{A}\mathcal{E}_{ab}\mathcal{D}^n \boldsymbol{\Psi}(z) = \frac{1}{(2\pi i)^2} \int_{z \circlearrowleft 0} dz \int_{u \circlearrowleft z} du \frac{\boldsymbol{\Psi}^*(u)\mathcal{E}_{ab}\mathcal{D}^n\boldsymbol{\Psi}(z)}{u-z}.$$

Summarizing the statements above we establish the operator $\mathrm{T}_{ab,n}$ as the pullback of the Yangian generator $t_{ab,n}$ in $\Lambda^{s,N}_-$.

Proposition 6.2 *For any $|v\rangle \in \mathcal{F}^s$ and $N \in \mathbb{N}$ we have the equality*

$$\pi_N(\mathrm{T}_{ab,n}|v\rangle) = t_{ab,n}\pi_N|v\rangle.$$

We note the importance of the polynomial property of the total zero mode in the constructed Yangian action on the Fock space $\mathcal{F}^s$, which we prove by using projective properties of the Yangian action in the phase spaces of CS models, it allows to formulate the following

Theorem 6.1 *The operators $\mathrm{T}_{ab,n}$ satisfy Yangian relations.*

In particular, the coefficients of the quantum determinant $q\det \mathrm{T}(u)$ form a commutative family which can be regarded as the limits of the higher Hamiltonians of CS system.

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1 Bosonic limit of Calogero-Sutherland system

1.1 Integrability of quantum Calogero-Sutherland model

Consider the quantum Calogero-Sutherland model of N particles on the circle [9, 23]. Its Hamiltonian is

$$H^{CS} = -\sum_{i=1}^{N} \left(\frac{\partial}{\partial q_i}\right)^2 + 2\left(\frac{\pi}{L}\right)^2 \sum_{i<j}^{N} \frac{\beta(\beta - K_{ij})}{\sin^2\left(\frac{\pi}{L}(q_i - q_j)\right)}, \tag{1.1}$$

where K_{ij} is the coordinate exchange operator of particles i and j.[1] After conjugating by the function $\prod_{i<j} |\sin(\frac{\pi}{L}(q_i - q_j))|^\beta$ which represents the vacuum state with eigenenergy $E_0 = (\pi\beta/L)^2 N(N^2-1)/3$, and passing to the exponential variables $x_i = e^{\frac{2\pi i q_i}{L}}$ we arrive after simple rescaling to the effective Hamiltonian

$$H^{eff} = \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i}\right)^2 + \beta \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\right) - 2\beta \sum_{i<j} \frac{x_i x_j}{(x_i - x_j)^2} (1 - K_{ij}). \tag{1.2}$$

We consider the symmetric and skewsymmetric wave functions of the Hamiltonian (1.2):

$$\phi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = \pm\phi(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N),$$

then the eigenfunctions of the Hamiltonian H^{CS}

$$\prod_{i<j} |\sin(q_i - q_j)|^\beta \phi(e^{2\pi i q_1}, \ldots, e^{2\pi i q_N})$$

are also (skew)symmetric by the variables $\{q_i\}$ (except for the vacuum state in skewsymmetric case). Denote by $\tilde{H}$ and $\bar{H}$ the restriction of the Hamiltonian (1.2) on the space of symmetric and skewsymmetric functions, respectively, then

$$\tilde{H} = \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i}\right)^2 + \beta \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\right); \tag{1.3}$$

$$\bar{H} = \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i}\right)^2 + \beta \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\right) - 4\beta \sum_{i<j} \frac{x_i x_j}{(x_i - x_j)^2}. \tag{1.4}$$

Further we use the Heckman–Dunkl operators $D_i^{(N)}$ in the form suggested by Polychronakos [46, 15]:

$$D_i^{(N)} = x_i \frac{\partial}{\partial x_i} + \beta \sum_{j \neq i} \frac{x_i}{x_i - x_j} (1 - K_{ij}). \tag{1.5}$$

These operators satisfy the relations

$$K_{ij} D_i^{(N)} = D_j^{(N)} K_{ij},$$
$$[D_i^{(N)}, D_j^{(N)}] = \beta(D_j^{(N)} - D_i^{(N)}) K_{ij},$$

[1] In literature one can find the Hamiltonian (1.1) in the form, where instead the exchange operator K_{ij} presents identity operator. In other notations it corresponds to the case of symmetric wave functions.

which coincide with the relations of the degenerate affine Hecke algebra after the renormalization $D_i^{(N)} \to \frac{1}{\beta} D_i^{(N)}$. We introduce the operators

$$\tilde{H}_k^{(N)} = \mathrm{Res}_+ \left(\sum_i \left(D_i^{(N)} \right)^k \right), \tag{1.6}$$

$$\bar{H}_k^{(N)} = \mathrm{Res}_- \left(\sum_i \left(D_i^{(N)} \right)^k \right), \tag{1.7}$$

where $\mathrm{Res}_\pm$ means the restriction on the space of symmetric and antisymmetric functions, respectively. As an example the first operators has the form

$$\tilde{H}_0^{(N)} = \bar{H}_0^{(N)} = N; \qquad \tilde{H}_1^{(N)} = \sum_{i=1}^N \left(x_i \frac{\partial}{\partial x_i} \right);$$

$$\bar{H}_1^{(N)} = \sum_{i=1}^N \left(x_i \frac{\partial}{\partial x_i} \right) + \beta N(N-1).$$

Proposition 1.1 *[17] i) The operators (1.6) commute.*
ii) The operators (1.7) commute.

Due to the Theorem 1.1 operators (1.6) and (1.7) can be chosen as the integrals of motion of the quantum Calogero-Sutherland model. In symmetric case the Hamiltonian (1.3) is given by

$$\tilde{H} = \tilde{H}_2^{(N)},$$

the expression of the Hamiltonian (1.4) in terms of $\bar{H}_k^{(N)}$ is given by the formula:

$$\bar{H} = \bar{H}_2^{(N)} - 2\beta(N-1)\bar{H}_1^{(N)} + \beta^2 N(N-1)^2. \tag{1.8}$$

1.2 Review of the scalar finite system

In this section we deal with the scalar CS system with N bosonic particles and review recent results [39, 52] mainly following the approach of [39]. The main idea is to regard the equivariant Heckman-Dunkl operators as a quantum L-operator acting on the space of polynomial functions of one variable with coefficients being symmetric polynomials of the remaining $N-1$ variables.

Due to the previous section the higher Hamiltonians of the system are expressed by means of Dunkl operators and can be chosen as any symmetric functions of $D_i^{(N)}$, as an example power sums. Clearly, symmetric functions of $D_i^{(N)}$ preserve the ring of symmetric polynomials $\Lambda_+^N = \mathbb{C}[x_1, \dots, x_N]^{S_N}$. This algebra is generated by the Newton polynomials $p_k^{(N)} = x_1^k + \dots + x_N^k$, $k \in 0, 1, \dots N$ (sometimes we omit the upper index N and simply write p_k). The Dunkl operator $D_i^{(N)}$ itself preserves the symmetry involving all variables other than x_i and therefore it acts on the space $\Lambda_+^{N,i}$ of functions symmetric in all variables except x_i:

$$\Lambda_+^{N,i} \backsimeq \mathbb{C}[x_i] \otimes \mathbb{C}[x_1, \dots x_{i-1}, x_{i+1}, \dots x_N]^{S_{N-1}}. \tag{1.9}$$

Denote by $\tilde{D}_i^{(N)}$ the restriction of the action of Dunkl operators on space $\Lambda_+^{N,i}$. Then the higher Hamiltonians (1.6) can be simply rewritten as:

$$\tilde{H}_k^{(N)} = \left(\sum_i \left(\tilde{D}_i^{(N)} \right)^k \right). \tag{1.10}$$

The action of Dunkl operator on symmetric function in N variables provides a family of N equivariant functions: $f_i(x_1, \dots ; x_i ; \dots x_N) \in \Lambda_+^{N,i}$ that

$$K_{ij} f_i = f_j.$$

For any $f(x_1, \dots ; x_i ; \dots x_N) \in \Lambda_+^{N,i}$ denote by $(\mathrm{E}_N\, f) \in \Lambda_+^N$ the following sum

$$\mathrm{E}_N\, f = \sum_{j=1}^N K_{ij} f.$$

For an equivariant family of functions it can be written as:

$$(\mathrm{E}_N\, f)(x_1, ..., x_N) = f_1(x_1; x_2, ..., x_N) + f_2(x_1; x_2; x_3, ..., x_N) + \dots \\ + f_N(x_1, ..., x_{N-1}; x_N).$$

An operator $\mathrm{E}_N : \Lambda_+^{N,i} \to \Lambda_+^N$ coincides up to a scalar factor with total symmetrization.

The action of the higher Hamiltonian (1.10) can be obtained by the following procedure: we start with symmetric fucntion $f(x_1, \dots, x_N) \in \Lambda_+^N$ and construct an equivariant family of its N copies using the natural embedding $\tilde{\iota}_{N,i} : \Lambda_+^N \to \Lambda_+^{N,i}$:

$$f_i(x_1, \dots ; x_i ; \dots x_N) = \tilde{\iota}_{N,i} f(x_1, \dots, x_N) \in \Lambda_+^{N,i}.$$

Then the action of Dunkl operator can be rewritten as

$$\tilde{D}_i^{(N)} f_i = x_i \frac{\partial}{\partial x_i} f_i + \beta \sum_{j \neq i} \frac{x_j}{x_i - x_j} (f_i - f_j).$$

We act the k-th power of Dunkl operator $(\tilde{D}_i^{(N)})^k$ and obtain the equivariant family:

$$\left((\tilde{D}_1^{(N)})^k f(x_1, \dots, x_N), \dots, (\tilde{D}_N^{(N)})^k f(x_1, \dots, x_N) \right),$$

then we symmetrize the answer using operator E_N. This procedure can be illustrated by the following matrix formula:

$$\tilde{H}_k = \begin{pmatrix} 1, 1, \dots \end{pmatrix} \begin{pmatrix} x_1 \frac{\partial}{\partial x_1} + \beta \sum\limits_{i=2}^N \dfrac{x_1}{x_1 - x_i} & -\beta \frac{x_1}{x_1 - x_2} & \dots \\ -\beta \frac{x_2}{x_2 - x_1} & x_2 \frac{\partial}{\partial x_2} + \beta \sum\limits_{i \neq 2} \dfrac{x_2}{x_2 - x_i} & \vdots \\ \vdots & \dots & \ddots \end{pmatrix}^k \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}$$

Our next aim to reformulate action of operators $\tilde{\imath}_{N,i}, \mathrm{E}_N, \tilde{D}_i^{(N)}$ in terms of Newton sums $p_k^{(N)}$. In the following we use the notation

$$V_+(z) = \exp\left(\sum_{n>0} z^n \frac{\partial}{\partial p_n}\right) \tag{1.11}$$

for the linear map, which changes each occurrence of a Newton sum $p_k^{(N)}$ by $p_k^{(N-1)} + z^k$.

Let $F = F(\{p_k^{(N)}\}) \in \Lambda_+^N$ be a symmetric function in N variables written in terms of $p_k^{(N)}$.

Lemma 1.1 *The natural embedding $\tilde{\imath}_{N,i} : \Lambda_+^N \to \Lambda_+^{N,i}$ is given by*

$$\tilde{\imath}_{N,i}(F) = V_+(x_i)F. \tag{1.12}$$

Here $V_+(x_i)F$ is a function of x_i and $\{p_k\}$ depending on $(N-1)$ variables.

Proof. The embedding $\tilde{\imath}_{N,i}$ can be regarded as the presentation of a symmetric function F by a polynomial in x_i with coefficients being symmetric functions of the remaining variables:

$$F(\{p_k\}) = F_0(x_1, \dots x_{i-1}, x_{i+1}, \dots x_N) + F_1(x_1, \dots x_{i-1}, x_{i+1}, \dots x_N)x_i +$$

$$F_2(x_1, \dots x_{i-1}, x_{i+1}, \dots x_N)x_i^2 + \dots .$$

This expansion can be obtained by means of a substitution

$$p_k^N \to p_k^{N-1} + x_i^k \tag{1.13}$$

which in its turn can be obtained by applying the vertex operator (1.11) due to the Taylor formula

$$f(z+t) = \exp\left(t\frac{\partial}{\partial z}\right) f(z) = f(z) + f'(z)t + \frac{1}{2}f''(z)t^2 + \dots$$

which yields a finite sum for polynomials. Observe that the formula (1.12) is correct for any expression of the symmetric function in terms of Newton polynomials p_k irrespective of their dependencies. Indeed, since both sides of (1.13) are equal as functions in $x_1, x_2, \dots, x_N$, the same is true for both sides of (1.12). ■

Let $\varphi^{\pm}(\xi)$ be the following power series in $\xi^{\pm 1}$:

$$\varphi^+(\xi) = \sum_{n=1}^{\infty} \xi^n \frac{n\partial}{\partial p_n}, \qquad \varphi^-(\xi) = \sum_{n=0}^{\infty} \frac{p_n}{\xi^n}, \tag{1.14}$$

where $p_0 = N$. We also use the notation

$$V'_+(z) = \exp\left(\sum_{n>0} -z^n \frac{\partial}{\partial p_n}\right). \tag{1.15}$$

By definition the operator $V'_+(z)$ changes each occurrence of the formal variable p_k^{N-1} by the difference $p_k^N - z^k$. The operator $V'_+(x_i)$ maps the space $\Lambda_+^{N,i}$ to $\Lambda_+^N \otimes \mathbb{C}[x_i]$. Note that

$$V'_+(z)V_+(z)F = F \qquad \forall\ F \in \Lambda_+^N. \tag{1.16}$$

Lemma 1.2 *Let $F(x_i;\{p_n\}) \in \Lambda_+^{N,i}$. Then*

$$\mathrm{E}_N\, F(\{p_n\}) = \oint \frac{d\xi}{\xi}\varphi^-(\xi)\left(V'_+(\xi)F\right)(\xi;\{p_n\}). \tag{1.17}$$

Here on the RHS the function $F(\xi;\{p_n\})$ depends on symmetric functions of $(N-1)$ variables, while $\mathrm{E}_N\, F(\{p_n\})$ on the LHS and $V'_+(\xi)F$ on the RHS both depend on symmetric functions on N variables. The integral on the right hand side counts the residue at infinity:

$$\oint f(\xi)d\xi = f_{-1} \text{ for } f(\xi) = \sum_i f_i\xi^i.$$

Proof. Due to

$$\oint \frac{d\xi}{\xi}\varphi^-(\xi)\xi^n = p_n$$

the formal integral on the RHS (1.17) changes each ξ^n in $V'_+(\xi)F$ to p_n, which coincides with the action of operator E_N. ■

Lemma 1.3 *The action of the Dunkl operator $\tilde{D}_i^{(N)}$ on functions $F(x_i,\{p_n\}) \in \Lambda_+^{N,i}$ can be expressed by the following formula:*

$$\begin{aligned}\tilde{D}_i^{(N)}(F(x_i,\{p_n\})) = x_i\frac{\partial}{\partial x_i}F(x_i,\{p_n\}) + &\\ + \beta x_i \oint \frac{d\xi}{\xi^2}\frac{\varphi^-(\xi)-1}{1-\frac{x_i}{\xi}}\left(V'_+(\xi)V_+(x_i)F\right)(\xi,\{p_n\}).\end{aligned} \tag{1.18}$$

Proof. In detail, the Dunkl operator $\tilde{D}_i^{(N)}$ transforms the space $\Lambda_+^{N,i}$ of functions with chosen variable x_i into itself:

$$\tilde{D}_i^{(N)}(F(x_i,\{p_n\})) = x_i\frac{\partial}{\partial x_i}F(x_i,\{p_n\}) + \beta\sum_{j\neq i}\frac{x_i}{x_i-x_j}(1-K_{ij})F(x_i,\{p_n\}) =$$

$$x_i\frac{\partial}{\partial x_i}F(x_i,\{p_n\}) + \beta\sum_{j\neq i}\frac{x_i}{x_i-x_j}\left(\left(V_+(x_j)F\right)(x_i,\{p_n\}) - \left(V_+(x_i)F\right)(x_j,\{p_n\})\right)$$

In each occurrence of $F(x_i;\{p_n\})$ we regard $\{p_n\}$ as symmetric functions of $(N-1)$ variables, while in $\left(V_+(x_j)F\right)(x_i,\{p_n\})$ $\{p_n\}$ depend on $(N-2)$ variables (all except x_i and x_j). Using the absence of singularities on the diagonals $x_i = x_j$ for Dunkl operators, we first present each fraction in the series as a function of x_i/x_j, then replace them by Cauchy integrals, to get:

$$\begin{aligned}\tilde{D}_i^{(N)}(F(x_i,\{p_n\})) &= x_i\frac{\partial}{\partial x_i}F(x_i,\{p_n\})\\ &-\beta\sum_{j\neq i}\frac{\frac{x_i}{x_j}}{1-\frac{x_i}{x_j}}\left(\left(V_+(x_j)F\right)(x_i,\{p_n\}) - \left(V_+(x_i)F\right)(x_j,\{p_n\})\right)\\ &= x_i\frac{\partial}{\partial x_i}F(x_i,\{p_n\})\\ &+\beta x_i\oint\frac{d\xi}{\xi^2}\frac{\varphi^-(\xi)-1}{1-\frac{x_i}{\xi}}\left(V'_+(\xi)V_+(x_i)F\right)(\xi,\{p_n\})\\ &-\beta x_i\oint\frac{d\xi}{\xi^2}\frac{\varphi^-(\xi)-1}{1-\frac{x_i}{\xi}}\left(V'_+(\xi)V_+(\xi)F\right)(x_i,\{p_n\}).\end{aligned}$$

In the last summand the vertex operators cancel each other due to (1.16), and the corresponding integral vanishes since it contains ξ only in negative powers. We then obtain formula (1.18). ∎

1.3 Bosonic limit in the extended ring of symmetric functions $\hat{\Lambda}$

The ring Λ of symmetric functions with infinite number of variables is defined as the projective limit $\Lambda = \varprojlim \Lambda_+^N$ with respect to the projection $\Lambda_+^{N+1} \to \Lambda_+^N$ [30, II.2]:

$$f(x_1, x_2, \ldots, x_N, x_{N+1}) \to f(x_1, x_2, \ldots, x_N, 0).$$

An element of Λ can be represented by a sequence of symmetric functions:

$$f_1(x_1), f_2(x_1, x_2), \ldots, f_N(x_1, x_2, \ldots, x_N), \ldots, \tag{1.19}$$

that stabilizes $f_{N+1}(x_1, x_2, \ldots, x_N, 0) = f_N(x_1, x_2, \ldots, x_N)$.

The ring Λ_+^N is generated by Newton power sums $p_k^{(N)}(x_1, x_2, \ldots, x_N) = \sum_{i=1}^N x_i^k$ $(k \leqslant N)$. The Newton polynomials satisfy the stability condition (1.19) and thus correctly define an element $p_k \in \Lambda$ that can be presented as a series $p_k = \sum_i x_i^k$. The elements p_k, $k = 0, 1, \ldots$ freely generate the ring Λ.

We add to Λ the formal variable p_0 and denote by $\hat{\Lambda} = \mathbb{C}[p_0, p_1, \ldots]$ the ring of symmetric functions extended by the free variable p_0 . The canonical projection $\tilde{\pi}_N : \hat{\Lambda} \to \Lambda_+^N$ can be desribed by the relation:

$$\tilde{\pi}_N : \hat{\Lambda} \to \Lambda_+^N : \quad p_k \to p_k^{(N)} = \sum_{i=1}^N x_i^k, \quad p_0 \to N. \tag{1.20}$$

The space $\hat{\Lambda}$ is an irreducible representation of the Heisenberg algebra $\mathcal{H}$, generated by the elements p_n and $\dfrac{\partial}{\partial p_n}$ and can be regarded as a polynomial version of the Fock space. It contains the vacuum vector $|0\rangle_+$, such that

$$\frac{\partial}{\partial p_n}|0\rangle_+ = 0, \qquad n = 0, 1, \ldots.$$

The dual vacuum vector ${}_+\langle 0|$ satisfies the condition

$${}_+\langle 0|p_n = 0, \qquad n = 0, 1, \ldots.$$

Introduce an operator $\Phi(z) : \hat{\Lambda} \otimes \mathbb{C}[z] \to \hat{\Lambda} \otimes \mathbb{C}[z]$

$$\Phi(z) = \exp\left(\sum_{n \geqslant 0} z^n \frac{\partial}{\partial p_n}\right). \tag{1.21}$$

In these terms the projection $\tilde{\pi}_N : \hat{\Lambda} \to \Lambda_+^N$ can be defined for an element $|v\rangle_+ \in \hat{\Lambda}$ as the following matrix element:

$$\tilde{\pi}_N |v\rangle_+ = {}_+\langle 0|\Phi(x_N) \ldots \Phi(x_2)\Phi(x_1)|v\rangle_+. \tag{1.22}$$

Indeed, for any $|v\rangle_+ = F(p_0, p_1, p_2, \dots)$ the operator $\Phi(x_i)$ shifts each p_n by x_i^n, so we have

$$\Phi(x_N)\dots\Phi(x_2)\Phi(x_1)|v\rangle_+ = F(p_0+N, p_1+\sum_{i=1}^N x_i, p_2+\sum_{i=1}^N x_i^2, \dots).$$

The left vacuum ${}_+\langle 0|$ "kills" each presence of p_n, so we obtain

$$\tilde{\pi}_N F(p_0, p_1, p_2, \dots) = F(N, \sum_{i=1}^N x_i, \sum_{i=1}^N x_i^2, \dots),$$

which coincides with the definition (1.20).

Lemma 1.4 *We have the following equality of linear maps $\hat{\Lambda} \to \Lambda_+^{N,i}$:*

$$\tilde{\pi}_{N-1}\Phi(x_i) = \tilde{\imath}_{N,i}\tilde{\pi}_N. \tag{1.23}$$

Proof. Applying both sides of of (1.23) to an element $|v\rangle_+ \in \hat{\Lambda}$ we get the tautology: both sides are equal to

$${}_+\langle 0|\Phi(x_N)\dots\Phi(x_2)\Phi(x_1)|v\rangle_+,$$

since $\Phi(x_i)$ and $\Phi(x_j)$ commute. ■

Introduce an operator $\Phi^*(z) : \hat{\Lambda}\otimes\mathbb{C}[z] \to \hat{\Lambda}\otimes\mathbb{C}[z]$:

$$\Phi^*(z) = \varphi^-(z)\exp\left(-\sum_{n\geqslant 0} z^n \frac{\partial}{\partial p_n}\right), \tag{1.24}$$

where $\varphi^-(z)$ is defined in (1.14). Define a linear map $\mathcal{S} : \hat{\Lambda}\otimes\mathbb{C}[z] \to \hat{\Lambda}$ as

$$\mathcal{S}F(\{p_n\}) = \oint \frac{d\xi}{\xi}\Phi^*(\xi)F(\xi, \{p_n\}).$$

The following lemma establishes the map $\mathcal{S}$ as the pullback of the finite symmetrization.

Lemma 1.5 *For each $F(z, \{p_n\}) \in \mathbb{C}[z]\otimes\hat{\Lambda}$ and any natural N we have the equality of elements of Λ_+^N :*

$$\mathrm{E}_N\,\tilde{\pi}_{N-1}F(z, \{p_n\}) = \tilde{\pi}_N\mathcal{S}(F(z, \{p_n\}). \tag{1.25}$$

Proof. The RHS of (1.25) equals

$${}_+\langle 0|\Phi(x_N)\dots\Phi(x_2)\Phi(x_1)\oint\frac{d\xi}{\xi}\Phi^*(\xi)F(\xi, p_0, p_1, p_2, \dots) =$$

$${}_+\langle 0|\Phi(x_N)\dots\Phi(x_2)\Phi(x_1)\oint\frac{d\xi}{\xi}\left(N+\sum_k \frac{p_k}{\xi^k}\right)V'(\xi)F(\xi, p_0-1, p_1, p_2, \dots).$$

The last is equal to LHS of (1.25) due to (1.17). Here we use a difference between the definitions (1.15),(1.14) and (1.24) for $\varphi^-(x)V_+'(x)$ and $\Phi^*(z)$ in the zero mode p_0 and $\frac{\partial}{\partial p_0}$.

■

Define an operator $\tilde{D} : \hat{\Lambda}\otimes\mathbb{C}[z] \to \hat{\Lambda}\otimes\mathbb{C}[z]$

$$\tilde{D}(F(z, \{p_n\})) = z\frac{\partial}{\partial z}F(z, \{p_n\}) + \beta z\oint\frac{d\xi}{\xi^2}\frac{1}{1-\frac{z}{\xi}}\Phi^*(\xi)\Phi(z)F(\xi, \{p_n\}). \tag{1.26}$$

The main result of this section we formulate as the following

Theorem 1.1 *The operator $\tilde{D}$ (1.26) is a limit of Dunkl operators $\tilde{D}_i^{(N)}$.*

In other words, the operator $\tilde{D}$ is a pullback of $\tilde{D}_i^{(N)}$ under the map $\tilde{\pi}_N$, defined in (1.22). We illustrate it by the following commutative diagram:

$$\begin{array}{ccc} \hat{\Lambda}\otimes\mathbb{C}[z] & \xrightarrow{\tilde{\pi}_{N-1}} & \Lambda_+^{N,i} \\ \downarrow{\scriptstyle \tilde{D}} & & \downarrow{\scriptstyle \tilde{D}_i^{(N)}} \\ \hat{\Lambda}\otimes\mathbb{C}[z] & \xrightarrow[\tilde{\pi}_{N-1}]{} & \Lambda_+^{N,i} \end{array} \tag{1.27}$$

The commutativity of (1.27) follows from Lemma 1.5 and formulas (1.18) and (1.26).
Comment. This result was formulated in other terms by Nazarov and Sklyanin, Sergeev and Veselov, see [39] and [52].

Theorem 1.1 implies the following

Proposition 1.2 *The operators $\tilde{\mathscr{H}}_k = \mathcal{S}\tilde{D}^k\tilde{\iota} : \hat{\Lambda} \to \hat{\Lambda}$,*

$$\tilde{\mathscr{H}}_k : \hat{\Lambda} \xrightarrow{\tilde{\iota}} \hat{\Lambda}\otimes\mathbb{C}[z] \xrightarrow{D^k} \hat{\Lambda}\otimes\mathbb{C}[z] \xrightarrow{\mathcal{S}} \hat{\Lambda}, \tag{1.28}$$

generate a commutative family of Hamiltonians of the limiting system [39, 52].

Proof. The operators $\tilde{\mathscr{H}}_k$ are the pullbacks of $\tilde{H}_k^{(N)}$ under the map $\tilde{\pi}_N$ due to Theorem 1.1 , Lemma 1.5 and (1.23). Due to the property

$$\cap_{N\in\mathbb{N}} \operatorname{Ker} \tilde{\pi}_N = 0 \tag{1.29}$$

of the ring of symmetric functions the commutativity of $\tilde{H}_k^{(N)}$ for $N \in \mathbb{N}$ implies the commutativity of $\tilde{\mathscr{H}}_k$. ■

As an example let us calculate the first Hamiltonians $\tilde{\mathscr{H}}_1$ and $\tilde{\mathscr{H}}_2$:

$$\tilde{\mathscr{H}}_1 = \oint \frac{d\xi}{\xi}\Phi^*(\xi)\left(\xi\frac{\partial}{\partial\xi}\right)\Phi(\xi) = \oint \frac{d\xi}{\xi}\varphi^-(\xi)\varphi^+(\xi) = \sum_{n>0} np_n\frac{\partial}{\partial p_n} \tag{1.30}$$

$$\tilde{\mathscr{H}}_2 = \oint \frac{d\xi}{\xi}\Phi^*(\xi)\left(\xi\frac{\partial}{\partial\xi}\right)^2\Phi(\xi) + +\beta\oint_{\eta<\xi}\frac{d\xi d\eta}{\xi^2}\Phi^*(\eta)\frac{1}{1-\frac{\eta}{\xi}}\Phi^*(\xi)\Phi(\eta)\left(\xi\frac{\partial}{\partial\xi}\right)\Phi(\xi) =$$

$$= \oint \frac{d\xi}{\xi}\varphi^-(\xi)\left(\varphi^+(\xi)\varphi^+(\xi) + \xi\frac{\partial}{\partial\xi}\varphi^+(\xi)\right) + \beta\oint_{\eta<\xi}\frac{d\xi d\eta}{\xi^2}\varphi^-(\eta)\left(\frac{\varphi^-(\xi)}{1-\frac{\eta}{\xi}} - \frac{1}{(1-\frac{\eta}{\xi})^2}\right)\varphi^+(\xi).$$

We have taken the second derivative in the first integral and used the commutator relations

$$[\varphi^-(\xi), \Phi(\eta)] = -\frac{\Phi(\eta)}{(1-\frac{\eta}{\xi})}$$

in the second. Thus

$$\tilde{\mathscr{H}}_2 = \oint \frac{d\xi}{\xi}\left(\varphi^-(\xi)\varphi^+(\xi)\varphi^+(\xi) + \beta(\varphi^-(\xi)-p_0)(\varphi^-(\xi)\varphi^+(\xi)) + (1-\beta)\varphi_-(\xi)\xi\frac{\partial}{\partial\xi}\varphi^+(\xi)\right)$$
$$= \sum_{k>0,n>0} knp_{k+n}\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_n} + \beta\sum_{k>0,n\geqslant 0}(k+n)p_kp_n\frac{\partial}{\partial p_{k+n}} + (1-\beta)\sum_{n>0} n^2p_n\frac{\partial}{\partial p_n}. \tag{1.31}$$

If we put $p_0 = 0$ and $\beta = 1$ the expression for $\tilde{\mathscr{H}}_2$ coincides with the so called “cut-and-join” operator which has applications in the combinatorics of Hurwitz numbers.

1.4 Classical limit and the Benjamin-Ono hierarchy

The Benjamin-Ono [8, 43] equation is nonlinear partial integro-differential equation appeared in hydrodynamics and describes one-dimensional internal waves in deep water:

$$u_t + 2uu_z + \gamma H(u_{zz}) = 0, \tag{1.32}$$

where

$$H(u(z)) = \frac{1}{\pi} p.v. \int dy \frac{u(y)}{y-z} = i(u^+(z) - u^-(z))$$

is the Hilbert transform. It is completely integrable [2, 24, 16] and has an infinite number of conserved integrals which are in involution with respect to Poisson bracket:

$$\{u(x), u(y)\} = \delta'(x-y).$$

The integrals of motion can be constructed recurrently using Backland (Miura) transformation [34].

The equation (1.32) can be rewritten in the form

$$u_t = \{u(z), \mathcal{I}\},$$

where the Hamiltonian $\mathcal{I}$ is defined by

$$\mathcal{I} = \int \left(\frac{1}{3} u^3(y) + \frac{\gamma}{2} \left(y \frac{\partial}{\partial y} u(y) \right) H(u(y)) \right) dy. \tag{1.33}$$

Introduce the following notation

$$\phi^+(z) = \varphi^+(z) = \sum_{n>0} n z^n \frac{\partial}{\partial p_n} \qquad \phi^-(z) = \beta(\varphi^-(z) - p_0) = \sum_{n>0} \frac{\beta p_n}{z^n}.$$

The commutation relations

$$[n \frac{\partial}{\partial p_n}, \beta p_k] = n\beta\delta_{n,k}$$

read

$$[\phi(x), \phi(y)] = \beta\delta'(x/y).$$

Consider a difference $\tilde{\mathscr{H}_2} - p_0 \tilde{\mathscr{H}_1}$ and multiply it by β, then using (1.30) and (1.31) we obtain

$$\tilde{\mathscr{H}} = \beta \left(\tilde{\mathscr{H}_2} - p_0 \tilde{\mathscr{H}_1} \right) = \frac{1}{3} \oint \frac{d\xi}{\xi} : \phi^3(\xi) : + \frac{1-\beta}{2} : \phi(\xi) \xi \frac{\partial}{\partial \xi} \left(\phi^+(\xi) - \phi^-(\xi) \right) :, \tag{1.34}$$

where : : means bosonic normal ordering: the operators p_n are moved to the left and the operators $\frac{\partial}{\partial p_n}$ are moved to the right. This leads to the following

Proposition 1.3 *The operator (1.33) with $\gamma = 1$ is the classical limit of the Hamiltonian (1.34) ($\beta \to 0$). The rule between the quantum commutator and Poisson bracket is $\beta^{-1}[\ ,\ ] \to \{\ \}$, see [39].*

Due to Proposition 1.3 one called the hierarchy (1.28) the quantum Benjamin-Ono hierarchy.

2 Fermionic limit

2.1 Polynomial phase space. Review of the finite system.

1. We regard the CS system of N fermionic particles with polynomial wave functions using the Heckman-Dunkl operators. The corresponding Heckman–Dunkl operators $D_i^{(N)}$: $\mathbb{C}[x_1,\dots,x_N] \to \mathbb{C}[x_1,\dots,x_N]$ are defined by the relation (1.5). Symmetric polynomials in $D_i^{(N)}$ preserve the space of symmetric Λ_+^N and antisymmetric polynomials Λ_-^N. Denote by $\alpha_N : \Lambda_+^N \to \Lambda_-^N$ the canonical isomorphism

$$\alpha_N : f(x_1,\dots x_N) \to \bar{f}(x_1,\dots x_N) = f(x_1,\dots x_N)\Delta(x_1,\dots,x_N), \tag{2.1}$$

where

$$\Delta(x_1,\dots,x_N) = \det_{i,j=1\dots N}(x_i^{N-j}) = \prod_{i<j}(x_i - x_j)$$

is the Vandermonde determinant.

The space Λ_+^N is generated by the Newton polynomials $p_k^{(N)} = x_1^k + \cdots + x_N^k$, $k = 1,\dots N$. Due to (2.1) any antisymmetric polynomial can be written by the following formula

$$\bar{f}(x_1 \dots x_N) = \Delta(x_1,\dots,x_N) f(\{p_k^{(N)}\}), \qquad k = 1,2,\dots$$

where f is a polynomial in p_k. Here and further we denote by $f(x_1,\dots x_N)$ or $f(\{p_k^{(N)}\}$ a symmetric function and by $\bar{f}(x_1,\dots x_N)$ the corresponding antisymmetric function following (2.1). For an operator B acting on the symmetric functions we denote by $\bar{\mathrm{B}}$ the corresponding operator acting on the antisymmetric functions so that the relation $\bar{\mathrm{B}}\bar{f}(x_1,\dots x_N) = \overline{\mathrm{B}\, f(x_1,\dots x_N)}$ holds.

The Dunkl operator $D_i^{(N)}$ preserves the antisymmetry involving all variables other than x_i. Denote by $\bar{D}_i^{(N)}$ the restriction of $D_i^{(N)}$ to the space of functions

$$\bar{f}(x_i; x_1,\dots,x_N) \in \Lambda_-^{N,i} \tag{2.2}$$

antisymmetric in all variables other than x_i. Due to (2.1) the LHS of (2.2) can be presented as

$$\bar{f}(x_i; x_1,\dots,x_N) = \Delta(x_1,\dots x_{i-1}, x_{i+1},\dots,x_N) f(x_i; \{p_k\}),$$

where f is a polynomial in x_i and in p_k, which depend on $N-1$ variables.

2. In the following we use the notations

$$V_+(z) = \exp\left(\sum_{n>0} z^n \frac{\partial}{\partial p_n}\right), \qquad V_-(z) = z^N \exp\left(-\sum_{n>0} \frac{p_n}{n z^n}\right), \tag{2.3}$$

where N is the number of variables in p_k. The operator $V_+(z)$ we introduced earlier in (1.11). More precisely, the operator $V_+(z)$ maps a polynomial expression in $\{p_k\}$ and in z into the same expression changing each occurrence of a Newton sum $p_k^{(N)}$ by $p_k^{(N-1)} + z^k$ due to the Taylor formula. The operator $V_-(z)$ does not change the number of variables in $p_k = p_k^{(N)}$ and can be equivalently written as an operator of multiplication by $\prod_i (z - x_i) \in \mathbb{C}[z] \otimes \Lambda_+^N$:

$$V_-(z) = z^N \prod_{i=1}^N \exp\left(-\sum_{n>0} \frac{x_i^n}{n z^n}\right) = z^N \prod_{i=1}^N \exp\left(\ln\left(1 - \frac{x_i}{z}\right)\right) = \prod_{i=1}^N (z - x_i). \tag{2.4}$$

Note that further we mostly use the composition of operators $V_-(z)V_+(z)$, which maps the space $\Lambda^{(+)}[z, x_2, \ldots, x_N]$ to $\mathbb{C}[z] \otimes \Lambda^{(+)}[x_2, \ldots, x_N]$. In this composition the operator $V_-(z)$ has the form $V_-(z) = z^{N-1} \exp\left(-\sum_{n>0} \frac{p_n}{nz^n}\right)$, where p_k depend on $N-1$ variables.

3. Let $f(\{p_k\})$ be a symmetric polynomial in N variables and

$$\bar{f}(x_1 \ldots x_N) = \Delta(x_1, \ldots, x_N) f(\{p_k\})$$

the corresponding antisymmetric polynomial. Denote by

$$\bar{\iota}_{N,i} : \Lambda_-^N :\to \Lambda_-^{N,i}$$

the natural embedding representing any antisymmetric polynomial as a polynomial in x_i with coefficients in $\Lambda^{(-)}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_N]$.

Proposition 2.1 *The embedding $\bar{\iota}_{N,i}$ is given by the following relation:*

$$\begin{aligned} \bar{\iota}_{N,i}(\bar{f}(x_1 \ldots x_N)) &= (-1)^{i+1}\overline{\iota_{N,i} f(\{p_k\})} = (-1)^{i+1}\overline{V_-(x_i)V_+(x_i)f(\{p_k\})} = \\ &= (-1)^{i+1}\Delta(x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_N)V_-(x_i)V_+(x_i)f(\{p_k\}). \end{aligned} \tag{2.5}$$

Here $V_-(x_i)V_+(x_i)f(\{p_k\})$ is a polynomial in x_i and in Newton polynomials $\{p_k\}$ depending on $(N-1)$ variables.

Proof. Using the definition of $\bar{\iota}_{N,i}$ we present the antisymmetric function $\bar{f}(x_1 \ldots x_N)$ in the following form

$$\begin{aligned} \bar{\iota}_{N,i}(\bar{f}(x_1 \ldots x_N)) =& \bar{f}_0(x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_N) + \bar{f}_1(x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_N)x_i + \\ &+ \bar{f}_2(x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_N)x_i^2 + \ldots, \end{aligned} \tag{2.6}$$

where each $\bar{f}_l(x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_N)$ is an antisymmetric polynomial. The decomposition (2.6) consists of two steps. The first one is a substitution

$$p_n^{(N)} \to p_n^{(N-1)} + x_i^n$$

in all the functions $f(\{p_k\})$, which is performed by the Taylor expansion

$$f(z+t) = \exp\left(t\frac{\partial}{\partial z}\right) f(z) = f(z) + f'(z)t + \frac{1}{2}f''(z)t^2 + \ldots$$

giving a finite sum for polynomials. The second step is a factorization of the Vandermonde determinant:

$$\Delta(x_1, \ldots, x_N) = \Delta(x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_N)(-1)^{i+1}\prod_{j\neq i}(x_i - x_j).$$

Due to (2.4) the factor $\prod_{j\neq i}(x_i - x_j)$ can be implemented in terms of p_k by applying the operator $V_-(x_i)$. Thus we obtain (2.5). ■

Observe that the formula (2.5) is correct for any expression of the symmetric function in terms of Newton polynomials p_k irrespective of their dependencies. Indeed,

$$V_-(z) = \prod_{i=1}^{N}(z - x_i) = \sum_{k\geq 0}^{N} e_k(x_1, \ldots, x_N)z^k,$$

where $e_k(x_1,\dots,x_N) = \sum_{1\le i_1<\dots<i_k\le N} x_{i_1}x_{i_2}\dots x_{i_k}$ are the elementary symmetric polynomials. They can be expressed by Newton sums $p_k(x_1,\dots,x_N)$ using Newton identities, and these expressions do not depend on the number of variables N.

4. We also use the notations

$$V'_+(z) = \exp\left(-\sum_{n>0} z^n \frac{\partial}{\partial p_n}\right), \qquad V'_-(z) = z^{-N}\exp\left(\sum_{n>0}\frac{p_n}{nz^n}\right). \tag{2.7}$$

By definition the operator $V'_+(z)$ changes each occurrence of the formal variable $p_k^{(N-1)}(x_1,\dots,x_{N-1})$ by the difference $p_k^{(N)}(x_1,\dots,x_{N-1},z)-z^k$. Thus the operator $V'_+(z)$ maps the space $\mathbb{C}[z]\otimes\Lambda^{(+)}[x_1,\dots,x_{N-1}]$ into itself, changing the meaning of the variables p_k. The operator $V'_-(z)$ can be equivalently written

$$V'_-(z) = z^{-N}\prod_{i=1}^N \exp\left(\sum_{n>0}\frac{x_i^n}{nz^n}\right) = z^{-N}\prod_{i=1}^N \exp\left(-\ln\left(1-\frac{x_i}{z}\right)\right) = z^{-N}\prod_i \frac{1}{(1-\frac{x_i}{z})} =$$

$$= z^{-N}\sum_{k\ge 0}\left(\sum_{1\le i_1\le i_2\le\dots\le i_k\le N} x_{i_1}x_{i_2}\dots x_{i_N}\right) z^{-k} = \sum_{k\ge0} h_k(x_1,\dots,x_N) z^{-k-N},$$

where $h_k(x_1,\dots,x_N) = \sum_{1\le i_1\le i_2\le\dots\le i_k\le N} x_{i_1}x_{i_2}\dots x_{i_N}$ are complete homogeneous symmetric polynomials. We then can rewrite

$$V'_-(z) = \sum_{k\ge0} h_k(\{p_n\}) z^{-k-N}, \tag{2.8}$$

where $h_k(\{p_n\})$ means that complete homogeneous symmetric polynomials are expressed from the basis of the Newton polynomials. These expressions do not depend on the number of variables N. So the operator $V'_-(z)$ transforms the space of polynomials in $p_k^{(N)}$ and in z into Laurent series in z with coefficients being polynomials in $p_k^{(N)}$.

5. Acting on antisymmetric function in N variables the Dunkl operators produce an equivariant family of N functions

$$\bar f_1(x_1;x_2,\dots,x_N), \qquad \bar f_2(x_2;x_1,x_3,\dots,x_N), \qquad \bar f_N(x_N;x_1,\dots,x_{N-1}),$$

where $\bar f_i(x_i;x_1,\dots,x_N)\in\Lambda_-^{N,i}$ and $K_{ij}\bar f_j(x_j;x_1,\dots,x_N) = -\bar f_i(x_i;x_1,\dots,x_N)$.

For any polynomial $\bar f(x_i;x_1,\dots,x_N)\in\Lambda_-^{N,i}$ denote by $\bar{\mathrm{A}}_N\bar f\in\Lambda_-^N$ the sum

$$(\bar{\mathrm{A}}_N\bar f)(x_1,\dots,x_N) = \bar f(x_1;x_2,\dots,x_N) - \bar f(x_2;x_1,x_3,\dots,x_N) - \dots - \bar f(x_N;x_1,\dots,x_{N-1}),$$

which we call the total antisymmetrization of the function $\bar f_i(x_i;x_1,\dots,x_N)$.

Let $f(x_i;\{p_k\})\in\Lambda_+^{N,i}$ and $\bar f(x_i;x_1,\dots,x_N)$ be the corresponding element of the space $\Lambda_-^{N,i}$:

$$\bar f(x_i;x_1,\dots,x_N) = (-1)^{i+1}\Delta(x_1,\dots x_{i-1},x_{i+1},\dots,x_N) f(x_i;\{p_k\}). \tag{2.9}$$

Proposition 2.2 *The total antisymmetrization* $(\bar{A}_N \bar{f})(x_1, ..., x_N)$ *can be described by the relation*

$$(\bar{\mathrm{A}}_N \bar{f})(x_1, ..., x_N) = \Delta(x_1, \ldots, x_N) \oint dz V'_-(z) V'_+(z) f(z; \{p_k\}) \tag{2.10}$$

Equivalently,

$$(\mathrm{A}_N\, f)(\{p_k\}) = \oint dz V'_-(z) V'_+(z) f(z; \{p_k\}).$$

Here on the RHS the function $f(z; \{p_k\})$ is a polynomial in z and in p_k depending on $(N-1)$ variables, while $V'_-(z)V'_+(z)f(z;\{p_k\})$ is a Laurent series in z with coefficients being polynomials in p_k depending on N variables. The integral on the right hand side counts the residue at infinity:

$$\oint f(z)dz = f_{-1} \text{ for } f(z) = \sum_i f_i z^i.$$

The proof of Proposition 2.2 is based on the following statement.

Lemma 2.1 *The following relation is valid*

$$x_1^k \Delta(x_2, \ldots, x_N) - x_2^k \Delta(x_1, x_3, \ldots, x_N) + \cdots + (-1)^{N+1} x_N^k \Delta(x_1, \ldots, x_{N-1}) =$$
$$= \begin{cases} \Delta(x_1, x_2, \ldots, x_N) h_{k+1-N}(x_1, \ldots, x_N) & \text{for } k \geq N-1 \\ 0 & \text{for } 0 \leq k < N-1 \end{cases},$$

Here $h_k(x_1, \ldots, x_N) = \sum\limits_{1 \leq i_1 \leq \cdots \leq i_k \leq N} x_{i_1} x_{i_2} \ldots x_{i_N}$ are complete homogeneous symmetric polynomials.

Proof of Lemma 2.1. Weyl formula for Schur polynomials says

$$s_{(\lambda_1, \lambda_2, \ldots, \lambda_N)}(x_1, x_2, \ldots, x_N) = \det_{i,j=1\ldots N}(x_i^{\lambda_j + N - j}) / \Delta(x_1, \ldots, x_N).$$

In particular, for $h_k(x_1, \ldots, x_N) = s_{(k,0,0,\ldots)}(x_1, \ldots, x_N)$ we have

$$h_{k+1-N}(x_1, ..., x_N) \Delta(x_1, x_2, \ldots, x_N) = \det \begin{pmatrix} x_1^k & x_2^k & \ldots & x_N^k \\ x_1^{N-2} & x_2^{N-2} & \ldots & x_N^{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \ldots & x_N \\ 1 & 1 & \ldots & 1 \end{pmatrix}. \tag{2.11}$$

For $0 \leq k < N-1$ the determinant in RHS of (2.11) equals zero. The statement of lemma now follows from (2.11) by the determinant expansion along the first row. See [54, \$ 7]. ∎

Proof of Proposition 2.2. Rewrite the relation (2.9) in the form

$$\bar{f}(x_i; x_1, \ldots, x_N) = (-1)^{i+1} \Delta(x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_N) f'(x_i; \{p_k\}),$$

where $f'(x_i; \{p_k\} = V'_+(x_i) f(x_i; \{p_k\})$ and p_k depends on N variables. The function $f'(x_i; \{p_k\}))$ is a polynomial in x_i and p_k:

$$f'(x_i; \{p_k\}) = \sum_{l=0}^{M} x_i^l f'_l(\{p_k\})$$

therefore we can realize antisymmetrization by each power of x_i independently:

$$(\bar{\mathrm{A}}_N\bar{f})(x_1,...,x_N) = \sum_{l=0}^{M} f_l^{'}(\{p_k\})\left(x_1^l\Delta(x_2,x_3,\ldots,x_N) - x_2^l\Delta(x_1,x_3,\ldots,x_N) + \ldots \right.$$
$$\left. +(-1)^{N+1}x_N^l\Delta(x_1,x_2,\ldots,x_{N-1})\right).$$

Due to Lemma 2.1

$$(\bar{\mathrm{A}}_N\bar{f})(x_1,...,x_N) = \Delta(x_1,x_2,...,x_N)\sum_{l=N-1}^{M} f_l^{'}(\{p_k\})h_{l+1-N}(x_1,...,x_N). \tag{2.12}$$

Due to (2.8) the formal integral

$$\oint dzV_-'(z)z^m = \begin{cases} h_{m+1-N}(\{p_n\}) & \text{for } m \geq N-1 \\ 0 & \text{for } 0 \leq m < N-1 \end{cases},$$

thus the integral $\oint dzV_-'(z)f^{'}(z;\{p_k\})$ gives the RHS of (2.12) divided by $\Delta(x_1,x_2,...,x_N)$. We then get (2.10). ■

6. Let $f(x_i;\{p_k\}) \in \Lambda_+^{N,i}$ and $\bar{f}(x_i;x_1,\ldots,x_N)$ be the corresponding element of the space $\Lambda_-^{N,i}$:

$$\bar{f}(x_i;x_1,\ldots,x_N) = (-1)^{i+1}\Delta(x_1,\ldots x_{i-1},x_{i+1},\ldots,x_N)f(x_i;\{p_k\}).$$

Define the operator

$$D_i^{(N)}:\Lambda_+^{N,i}\to\Lambda_+^{N,i}$$

by the relation

$$\begin{aligned} D_i^{(N)}f(x_i,\{p_k\}) &= x_i\frac{\partial}{\partial x_i}f(x_i,\{p_k\}) + \\ &\beta x_i\oint dz\frac{V_-'(z)V_+'(z)}{x_i - z}\left(V_-(z)V_+(z)f(x_i,\{p_k\}) - V_-(x_i)V_+(x_i)f\left(z,\{p_k\}\right)\right). \end{aligned} \tag{2.13}$$

Then we formulate the following:

Proposition 2.3 *The action of the Dunkl operator $\bar{D}_i^{(N)}$ in the space $\Lambda_-^{N,i}$ can be expressed by the relation:*

$$\begin{aligned} \bar{D}_i^{(N)}\ \bar{f}(x_i;x_1,\ldots,x_N) &= (-1)^{i+1}\overline{D_i^{(N)}f(x_i;\{p_k\})} = \\ &= (-1)^{i+1}\Delta(x_1,\ldots x_{i-1},x_{i+1},\ldots,x_N)D_i^{(N)}f(x_i;\{p_k\}). \end{aligned} \tag{2.14}$$

Proof. Firstly, we use the embedding $1\otimes\iota_{N,j}:\Lambda_+^{N,i}\to\Lambda_+^{N-2,i,j}$ from the proposition 2.1:

$$1\otimes\iota_{N,j}: f(x_i,\{p_n\})\to V_-(x_j)V_+(x_j)f(x_i,\{p_n\}).$$

Then the operator $\frac{x_i}{x_i-x_j}((1-K_{ij})$ can be written by the following formula

$$\begin{aligned} &\frac{x_i}{x_i-x_j}(1-K_{ij})V_-(x_j)V_+(x_j)f(x_i,\{p_n\}) = \\ &= \frac{x_i}{x_i-x_j}((V_-(x_j)V_+(x_j)f(x_i,\{p_n\}) - V_-(x_i)V_+(x_i)f\left(x_j,\{p_n\}\right)). \end{aligned} \tag{2.15}$$

Then we use the formula of total antisymmetrization from proposition 2.2. ■

7. Here we present the formulas for antisymmetrization in a form which we will use in the Fock space limit.

Remark 2.1 *The formal integral $\oint dz V'_-(z)V'_+(z)f(z;\{p_k\})$ for the polynomial $f(z;\{p_k\})$ in z can be rewritten as a complex integral*

$$\frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0} dz \int_{u\circlearrowleft z} du \frac{V'_-(u)V'_+(u)f(z,\{p_k\})}{u-z}. \tag{2.16}$$

Remark 2.2 *For $f(z;x_i;\{p_k\})$ with parameter x_i the formal integral for antisymmetrization $\oint dz V'_-(z)V'_+(z)f(z;x_i;\{p_k\})$ can be rewritten as*

$$\frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0, z\ll x_i} dz \int_{u\circlearrowleft z} du \frac{V'_-(u)V'_+(u)f(z;x_i;\{p_k\})}{u-z}. \tag{2.17}$$

Here we choose the countour so as to avoid the singularity $z = x_i$. This is a rule for how to use the composition of Dunkl operators.

8. To obtain the Hamiltonians

$$\bar{H}_k^{(N)} = \sum_i (\bar{D}_i^{(N)})^k$$

we replace the outer sum by antisymmetrization operator $\bar{\mathrm{A}}_N$ so that we get an expression which actually does not depend on i,

$$\bar{H}_k^{(N)} = \bar{\mathrm{A}}_N(\bar{D}_i^{(N)})^k \bar{\iota}_{N,i} = \overline{\mathrm{A}_N(D_i^{(N)})^k \iota_{N,i}}. \tag{2.18}$$

The procedure is illustrated by the following diagram

$$\Lambda_-^N \xrightarrow{\bar{\iota}_{N,i}} \Lambda_-^{N,i} \xrightarrow{(\bar{D}_i^{(N)})^k} \Lambda_-^{N,i} \xrightarrow{\bar{\mathrm{A}}_N} \Lambda_-^N \ .$$

The expressions for the first Hamiltonians $H_k^{(N)} = \left(\mathrm{A}_N(D_i^{(N)})^k \iota_{N,i}\right)$ are given below:

$$H_0^{(N)} = N,$$

$$H_1^{(N)} = \sum_{n>0} np_n \frac{\partial}{\partial p_n} + (1+2\beta)\,\frac{N^2-N}{2},$$

$$\begin{aligned} H_2^{(N)} = \sum_{n,k>0} nkp_{n+k}\frac{\partial}{\partial p_n}\frac{\partial}{\partial p_k} + (1+\beta)\sum_{n,k>0}(n+k)p_np_k\frac{\partial}{\partial p_{n+k}} - \beta\sum_{n>0} n^2p_n\frac{\partial}{\partial p_n} \\ - (1+2\beta)\sum_{n>0} np_n\frac{\partial}{\partial p_n} + (3\beta+2)N\sum_{n>0} np_n\frac{\partial}{\partial p_n} \\ +\frac{1}{6}(2N^3-3N^2+N) + \frac{\beta}{6}(7N^3-12N^2+5N) + \beta^2(N^3-2N^2+N). \end{aligned}$$

2.2 The limit in the space $\hat{\Lambda}$

1. Let $\hat{\Lambda} = \Lambda[p_0]$ be the ring of symmetric functions extended by the free variable p_0, defined at the beginning of section 1.3. Let $\Psi(z)$ and $\Psi^*(z)$ be vertex operators $\hat{\Lambda} \to \mathbb{C}[z, z^{-1}]] \otimes \hat{\Lambda}$,

$$\Psi(z) = z^{p_0} \exp\left(-\sum_{n>0} \frac{p_n}{nz^n}\right) \exp\left(\sum_{n\geq 0} z^n \frac{\partial}{\partial p_n}\right), \tag{2.19}$$

$$\Psi^*(z) = z^{-p_0} \exp\left(\sum_{n>0} \frac{p_n}{nz^n}\right) \exp\left(-\sum_{n\geq 0} z^n \frac{\partial}{\partial p_n}\right). \tag{2.20}$$

The following relations are valid:

$$\begin{aligned} \Psi(z)\Psi(w) &= (w-z) : \Psi(z)\Psi(w) : \\ \Psi(z)\Psi^*(w) &= \frac{1}{(w-z)} : \Psi(z)\Psi(w)^* :, \end{aligned} \tag{2.21}$$

where : : means bosonic normal ordering — all operators $\frac{\partial}{\partial p_n}$ are moved to the right and operators p_n are moved to the left. Operators (2.19) and (2.20) satisfy the relations:

$$\frac{1}{2\pi i} \int_{z \circlearrowleft w} \Psi(w)\Psi^*(z) dz = \frac{1}{2\pi i} \int_{z \circlearrowleft w} \Psi^*(w)\Psi(z) dz = 1.$$

2. Let $|v\rangle = f(p_0, p_1, ..., p_k, ...)|0\rangle \in \hat{\Lambda}$, where $f(p_0, p_1, ..., p_k, ...)$ is a polynomial in p_k. Define the evaluation map $\bar{\pi}_N : \hat{\Lambda} \to \Lambda_-^N$ by the prescription

$$\bar{\pi}_N|v\rangle = \langle 0|\Psi(x_N) \cdots \Psi(x_1)|v\rangle. \tag{2.22}$$

The function $\bar{\pi}_N|v\rangle$ is antisymmetric polynomial

$$\bar{\pi}_N|v\rangle = \prod_{i<j}(x_i - x_j) f(N, (x_1 + \ldots + x_N), ..., (x_1^k + \cdots + x_N^k), ...). \tag{2.23}$$

Indeed, $\Psi(x_N) \cdots \Psi(x_1) = \prod_{i<j}(x_i - x_j) : \Psi(x_N) \cdots \Psi(x_1) :$ due to (2.21). The operator $\prod_i \exp\left(\sum_{n\geq 0} x_i^n \frac{\partial}{\partial p_n}\right)$ replaces every item p_k in f with $x_1^k + \cdots + x_n^k$, $k = 0, 1, \ldots$, while

$$\langle 0|\prod_i x_i^{p_0} \exp\left(-\sum_{n>0} \frac{p_n}{nx_i^n}\right) = \langle 0|.$$

3. Similarly we define the map

$$\bar{\pi}_{N-1,i} : z^{p_0}\mathbb{C}[z, z^{-1}]] \otimes \hat{\Lambda} \to \mathbb{C}[x_i, x_i^{-1}]] \otimes \Lambda^{(-)}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_N]$$

as follows

$$\bar{\pi}_{N-1,i} : z^{p_0+k} \otimes |v\rangle \to (-1)^{i+1}\langle 0|\Psi(x_N) \cdots \Psi(x_{i+1})\Psi(x_{i-1}) \cdots \Psi(x_1) x_i^{p_0+k}|v\rangle. \tag{2.24}$$

Define the inclusion $\iota : \hat{\Lambda} \to z^{p_0}\mathbb{C}[z, z^{-1}]] \otimes \hat{\Lambda}$ by the relation

$$\iota(|v\rangle) = \Psi(z)|v\rangle.$$

Lemma 2.2 *The following diagram is commutative:*

$$\begin{tikzcd} \hat{\Lambda} \arrow[r, "\iota"] \arrow[d, "\bar{\pi}_N"'] & z^{p_0}\mathbb{C}[z,z^{-1}]]\otimes\hat{\Lambda} \arrow[d, "\bar{\pi}_{N-1,i}"] \\ \Lambda_-^N \arrow[r, "\bar{\iota}_{N,i}"'] & \Lambda_-^{N,i} \end{tikzcd} \tag{2.25}$$

Proof. Let us check the commutativity of the diagram (2.25) for the element $|v\rangle = f(p_0, p_1, ..., p_k, ...)|0\rangle \in \hat{\Lambda}$. The composition of $\bar{\pi}_N$ and $\bar{\iota}_{N,i}$ defines the natural embedding of the antisymmetric polynomial

$$\langle 0|\Psi(x_N)\cdots\Psi(x_1)|v\rangle$$

into the space $\Lambda_-^{N,i}$, which is the expansion of the function in $x_1, \ldots x_N$ over the variable x_i. Applying the maps ι and $\bar{\pi}_{N-1,i}$ we obtain the following relation

$$\bar{\pi}_{N-1,i}\iota|v\rangle = \bar{\pi}_{N-1,i}\Psi(z)|v\rangle =$$
$$(-1)^{i+1}\langle 0|\Psi(x_N)\cdots\Psi(x_{i+1})\Psi(x_{i-1})\cdots\Psi(x_1)\Psi(x_i)|v\rangle = \langle 0|\prod_{N\geq j\geq 1}\Psi(x_j)|v\rangle,$$

which coincides with natural embedding $\bar{\iota}_{N,i}$ of $\langle 0|\Psi(x_N)\cdots\Psi(x_1)|v\rangle$. ∎

4. Thus we have shown that for any $|v\rangle \in \hat{\Lambda}$ the element $\bar{\pi}_{N-1,i}\iota(|v\rangle) \subset \mathbb{C}[x_i] \otimes \Lambda^{(-)}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_N] = \Lambda_-^{N,i}$ is polynomial in x_i. Denote by U the space

$$U = \cap_N \bar{\pi}^{-1}_{N-1,i}\left(\Lambda_-^{N,i}\right).$$

Due to Lemma 2.2 we have the inclusion $\iota(\hat{\Lambda}) \subset U$.

Define the map $\mathcal{A} : z^{p_0}\mathbb{C}[z,z^{-1}]]\otimes\hat{\Lambda} \to \hat{\Lambda}$ of antisymmetrization as follows

$$\mathcal{A}F = \frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0} dz \int_{u\circlearrowleft z} du \frac{\Psi^*(u)F(z)}{u-z}, \tag{2.26}$$

where $F(z) \in z^{p_0}\mathbb{C}[z,z^{-1}]]\otimes\hat{\Lambda}$. In other words

$$\mathcal{A} : z^{p_0+k}\otimes|v\rangle \to \frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0} dz \int_{u\circlearrowleft z} du \frac{\Psi^*(u)z^{p_0+k}}{u-z}|v\rangle.$$

Lemma 2.3 *The following diagram is commutative:*

$$\begin{tikzcd} z^{p_0}\mathbb{C}[z,z^{-1}]]\otimes\hat{\Lambda} \supset U \arrow[r, "\mathcal{A}"] \arrow[d, "\bar{\pi}_{N-1,i}"'] & \hat{\Lambda} \arrow[d, "\bar{\pi}_N."] \\ \Lambda_-^{N,i} \arrow[r, "\bar{A}_N"'] & \Lambda_-^N \end{tikzcd} \tag{2.27}$$

Proof. We can present any element in U as a series $\sum_k z^{p_0+k}\otimes|v_k\rangle$. We check the commutativity of the diagram (2.27) for the element $z^{p_0+k}\otimes|v\rangle$, where $|v\rangle = f(p_0, p_1, ..., p_k, ...)|0\rangle$. Following the definitions we obtain:

$$\bar{\pi}_{N-1,i}(z^{p_0+k}\otimes|v\rangle) = (-1)^{i+1}\langle 0|\Psi(x_N)\cdots\Psi(x_{i+1})\Psi(x_{i-1})\cdots\Psi(x_1)x_i^{p_0+k}f(p_0,p_1,\dots)|0\rangle.$$

Thus

$$\bar{\pi}_{N-1,i}(z^{p_0+k}\otimes|v\rangle) = \Delta(x_1,\dots x_{i-1},x_{i+1},\dots,x_N)f(x_i;\{p_k\}),$$

where $f(z;\{p_k\}) = z^{k+N-1}f(N-1,p_1,p_2.\dots)$. Using Proposition 2.2 we obtain

$$\begin{aligned}
&\bar{\mathrm{A}}_N\bar{\pi}_{N-1,i}(z^{p_0+k}\otimes|v\rangle) = \\
&\qquad\frac{\Delta(x_1,\dots x_N)}{2\pi i}\int_{z\circlearrowleft 0} dz V'_-(z)V'_+(z)z^{k+N-1}f(N-1,p_1,p_2,\dots) = \\
&\qquad=\langle 0|\Psi(x_N)\dots\Psi(x_1)\frac{1}{2\pi i}\int_{z\circlearrowleft 0} dz V'_-(z)V'_+(z)z^{k+N-1}f(N-1,p_1,p_2\dots)|0\rangle.
\end{aligned}\tag{2.28}$$

Going by arrows $\bar{\pi}_N$ and $\mathcal{A}$ we get

$$\begin{aligned}
&\bar{\pi}_N\mathcal{A}(z^{p_0+k}\otimes|v\rangle) = \\
&\qquad\langle 0|\Psi(x_N)\cdots\Psi(x_1)\frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0}dz\int_{u\circlearrowleft z}du\frac{\Psi^*(u)z^{p_0+k}}{u-z}f(p_0,p_1,\dots|0\rangle.
\end{aligned}$$

To compare with the RHS of (2.28) we use the following transformations:

$$\begin{aligned}
&\bar{\pi}_N\mathcal{A}(z^{p_0+k}\otimes|v\rangle) = \\
&\langle 0|\Psi(x_N)\cdots\Psi(x_1)\frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0}dz\int_{u\circlearrowleft z}du\frac{V'_-(u)V'_+(u)e^{-\frac{\partial}{\partial p_0}}z^{p_0+k}}{u-z}f(p_0,p_1,\dots)|0\rangle = \\
&\langle 0|\Psi(x_N)\cdots\Psi(x_1)\frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0}dz\int_{u\circlearrowleft z}du\frac{V'_-(u)V'_+(u)z^{k+N-1}}{u-z}f(p_0-1,p_1,p_2,\dots)|0\rangle = \\
&\langle 0|\Psi(x_N)\dots\Psi(x_1)\frac{1}{2\pi i}\int_{z\circlearrowleft 0}dz V'_-(z)V'_+(z)z^{k+N-1}f(N-1,p_1,p_2,\dots)|0\rangle.
\end{aligned}$$

Thus we prove the commutativity of the diagram (2.27) for the element $z^{p_0+k}\otimes|v\rangle$. For the sum $\sum_k z^{p_0+k}\otimes|v_k\rangle$ we use the property of the space U, that its image by the projection $\bar{\pi}_{N-1,1}$ is a finite sum. ■

5. Define an operator $D:\hat{\Lambda}\otimes\mathbb{C}[z,z^{-1}]]\to\hat{\Lambda}\otimes\mathbb{C}[z,z^{-1}]]$

$$DF(z) = z\frac{\partial}{\partial z}F(z) + \beta\frac{1}{(2\pi i)^2}\int_{w\circlearrowleft 0}dw\int_{u\circlearrowleft w}\frac{du}{(u-w)}\frac{\Psi^*(u)}{\left(1-\frac{w}{z}\right)}\left(\Psi(w)F(z)-\Psi(z)F(w)\right).\tag{2.29}$$

Now we formulate the main result of this section

Theorem 2.1 *The operator D acting on the auxillary space U is a pullback of Heckman-Dunkl operators $\bar{D}_i^{(N)}$ under the map $\bar{\pi}_{N-1,i}$.*

Proof. Due to Lemmas 2.2, 2.3 we get the following commutative diagram:

$$\begin{tikzcd}
U\subset\hat{\Lambda}\otimes\mathbb{C}[z,z^{-1}]] \arrow[r, "\bar{\pi}_{N-1,i}"] \arrow[d, "D"] & \Lambda_-^{N,i} \arrow[d, "\bar{D}_i^{(N)}"] \\
U\subset\hat{\Lambda}\otimes\mathbb{C}[z,z^{-1}]] \arrow[r, "\bar{\pi}_{N-1,i}"'] & \Lambda_-^{N,i}
\end{tikzcd}\tag{2.30}$$

Due commutativity (2.30) D maps the subspace U into itself and is a pullback of $\bar{D}_i^{(N)}$ under the map $\bar{\pi}_{N-1,i}$. ■

6. Define operators $\mathscr{H}_k = \mathcal{A}D^k\iota : \hat{\Lambda} \to \hat{\Lambda}$ by the formula

$$\mathscr{H}_k : \hat{\Lambda} \xrightarrow{\iota} U \xrightarrow{D^k} U \xrightarrow{\mathcal{A}} \hat{\Lambda}. \tag{2.31}$$

Due to (2.30) we get the commutative diagram

$$\begin{tikzcd} \hat{\Lambda} \arrow[r, "\mathscr{H}_k"] \arrow[d, "\bar{\pi}_N"'] & \hat{\Lambda} \arrow[d, "\bar{\pi}_N"] \\ \Lambda_-^N \arrow[r, "\bar{H}_k^{(N)}"'] & \Lambda_-^N \end{tikzcd} \tag{2.32}$$

Proposition 2.4 *The operators $\mathscr{H}_k$ generate a commutative family of Hamiltonians of the limiting system.*

Proof. For any N operators $\bar{H}_k^{(N)}$ commute. Due to commutativity of (2.32) and the fact that $\cap Ker(\bar{\pi}_N) = \varnothing$ operators $\mathscr{H}_k$ commute as well. ■

We present the expression for the first Hamiltonians:

$$\mathscr{H}_0 = p_0,$$

$$\mathscr{H}_1 = \sum_{n>0} np_n \frac{\partial}{\partial p_n} + (1+2\beta)\,\frac{p_0^2 - p_0}{2},$$

$$\begin{aligned} \mathscr{H}_2 = &\sum_{n,k>0} nkp_{n+k}\frac{\partial}{\partial p_n}\frac{\partial}{\partial p_k} + (1+\beta)\sum_{\substack{n,k\geq 0\\ n+k>0}} (n+k)p_np_k\frac{\partial}{\partial p_{n+k}} \\ &-\beta\sum_{n>0} n^2p_n\frac{\partial}{\partial p_n} - (1+2\beta)\sum_{n>0} np_n\frac{\partial}{\partial p_n} + \beta p_0\sum_{n>0} np_n\frac{\partial}{\partial p_n} + \\ &+\frac{1}{6}(2p_0^3 - 3p_0^2 + p_0) + \frac{\beta}{6}(7p_0^3 - 12p_0^2 + 5p_0) + \beta^2(p_0^3 - 2p_0^2 + p_0). \end{aligned}$$

The limiting expression $\mathscr{H}$ corresponding to (1.4) can be expressed by the formula similar to (1.8):

$$\begin{aligned} \mathscr{H} = &\mathscr{H}_2 - 2\beta(p_0-1)\mathscr{H}_1 + \beta^2p_0(p_0-1)^2 = \\ = &\sum_{n,k>0} nkp_{n+k}\frac{\partial}{\partial p_n}\frac{\partial}{\partial p_k} + (1+\beta)\sum_{n>0,k\geq 0} (n+k)p_np_k\frac{\partial}{\partial p_{n+k}} - \beta\sum_{n>0} n^2p_n\frac{\partial}{\partial p_n} \\ &+(p_0-1)\sum_{n>0} np_n\frac{\partial}{\partial p_n} + \frac{1}{6}(2p_0^3 - 3p_0^2 + p_0) + \frac{\beta}{6}p_0(p_0^2 - 1). \end{aligned}$$

The Hamiltonian $\mathscr{H} + \mathscr{H}_1$ with shift $\beta \to (\beta - 1)$ coincides with the bosonic limiting expression (1.31) by putting $p_0 = 0$.

7 Comments. The space Λ of symmetric functions can be realized either as the projective limit of the rings of symmetric polynomials in N variables, or the projective limit of the spaces of antisymmetric polynomials in N variables. The latter means the commutativity of the diagrams

$$\begin{tikzcd} & \Lambda \arrow[dl, "\alpha_N"'] \arrow[dr, "\alpha_{N+1}"] & \\ \Lambda_-^N & & \Lambda_-^{N+1} \arrow[ll, "\omega^-_{N+1}"] \end{tikzcd},$$

where

$$\omega^-_{N+1} : \bar f(x_1, \dots x_N, x_{N+1}) \mapsto \bar f(x_1, \dots x_N, 0) \prod_{i=1}^{N} x_i^{-1}, \qquad \text{and}$$
$$\alpha_N : f(p_1, \cdots p_N) \mapsto \prod_{i<j}(x_i - x_j) f((x_1 + \ldots + x_N), ..., (x_1^k + \cdots + x_N^k), ...).$$

The space $\hat\Lambda$ is not a projective limit of the spaces Λ_-^N due to the presence of p_0 which breaks the commutativity of analogous diagram for $\hat\Lambda$ with α_N replaced by maps $\bar\pi_N$. On the other hand, CS Hamiltonians $\bar H_k$ theirselves do not compose the projective system since $\omega^-_{N+1}\bar H_k^{(N+1)} \neq \bar H_k^{(N)}\omega^-_{N+1}$. However, the Hamiltonians $\bar H_k^{(N)}$ written in form (2.18) are compatible with maps ω^-_{N+1}, if we replace each occurrence of N in $H_k^{(N)}$ to $N+1$ in $H_k^{(N+1)}$. Moreover, each finite Hamiltonian can be restored from its limit by formal replacement of each occurrence of p_0 by operator of multiplication on the number N of particles.

This correspondence hints the form of corrections in Hamiltonians to form a projective system: substract terms containing p_0 in the limit expression. Here are examples of corrections for the first Hamiltonians:

$$\begin{aligned} &H^{(N)}_{pr,1} = H^{(N)}_1 - (1+2\beta)\frac{N^2-N}{2}, \\ &H^{(N)}_{pr,2} = H^{(N)}_2 - 3\beta N H^{(N)}_{pr,1} \\ &-\frac{1}{6}(2N^3-3N^2+N) - \frac{\beta}{6}(7N^3-12N^2+5N) - \beta^2(N^3-2N^2+N). \end{aligned}$$

2.3 Realization in the Fock space

1. The constructed above Hamiltonians form a commutative family of operators in the space $\hat\Lambda$. Moreover, they commute inside the Heisenberg algebra and thus can be used as well in its other representations, for instance, in the bosonic Fock space $\mathcal{F}$. In this section we show how to realize the limit in the bosonic Fock space, the key point is to define the analog of projection $\bar\pi_N$. The formulas for the Hamiltonians remains the same.

The bosonic Fock space is usually defined as a free commutative algebra $\mathbb{C}[q, p_1, p_2, \dots]$ on varibles p_k and q. Define the vacuum vector $|0\rangle$ and a dual vacuum $\langle 0|$ of the bosonic Fock space $\mathcal{F}$:

$$\frac{\partial}{\partial p_n}|0\rangle = 0, \quad n \geq 1 \qquad \langle 0|p_n = 0, \qquad n \geq 0, \quad \langle 0|p_0 = p_0|0\rangle = 0.$$

Denote by $\langle n|$ and $|n\rangle$ the following vectors:

$$|n\rangle = e^{-n\frac{\partial}{\partial p_0}}|0\rangle = q^{-\ n}|0\rangle, \qquad \langle n| = \langle 0|q^n.$$

These vectors are biorthogonal $\langle n|m\rangle = \delta_{n,m}$ and have the following properties

$$\langle n|p_0 = n\langle n|, \quad p_0|n\rangle = n|n\rangle.$$

Any vector in space $\mathcal{F}$ can be presented as a linear combination of such vectors $|v\rangle = f(p_1, ..., p_k, ...)|c\rangle$, where $f(p_1, ..., p_k, ...)$ is a polynomial in p_k and c is so called charge of $|v\rangle$ and we denote it by $p_0(v)$. Denote by $\mathcal{F}_c$ the linear span of vectors with charge c, then $\mathcal{F}$ is graded according to the charge $\mathcal{F} = \oplus_{c\in\mathbb{Z}}\mathcal{F}_c$.

Define the projection $\pi_N : \mathcal{F} \to \Lambda_-^N$ by the prescription

$$\pi_N|v\rangle = \langle 0|\Psi(x_N)\cdots\Psi(x_1)|v\rangle. \tag{2.33}$$

Due to biorthogonality $\langle n|m\rangle = \delta_{n,m}$ and fact that product $\Psi(x_N)\cdots\Psi(x_1)$ contains q^N we have $\pi_N(\mathcal{F}_c) = 0$ for $c \neq N$. Thus for $|v\rangle = f(p_1, ..., p_k, ...)|c\rangle$ we have

$$\pi_N|v\rangle = \begin{cases} \prod_{i<j}(x_i - x_j) f((x_1 + \ldots + x_N), ..., (x_1^k + \cdots + x_N^k), ...) & \text{for } p_0(v) = N \\ 0 \quad \text{for } p_0(v) \neq N \end{cases}.$$

Similarly we define the map

$$\pi_{N-1,i} : z^{p_0}\mathbb{C}[z, z^{-1}]] \otimes \mathcal{F} \to \mathbb{C}[x_i, x_i^{-1}]] \otimes \Lambda^{(-)}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_N]$$

as follows

$$\begin{aligned}\pi_{N-1,i} :& z^{p_0+k} \otimes |v\rangle \to \\ & (-1)^{i+1}\langle 0|\Psi(x_N)\cdots\Psi(x_{i+1})\Psi(x_{i-1})\cdots\Psi(x_1)x_i^{p_0+k}|v\rangle.\end{aligned}$$

Due to the same arguments $\pi_{N-1,i}(z^{p_0}\mathbb{C}[z, z^{-1}]] \otimes \mathcal{F}_c) = 0$ if $c \neq N$. Then we have the analogous commutativity as in Lemma 2.2 for $\mathcal{F}$ and π_N instead of $\hat{\Lambda}$ and $\bar{\pi}_N$, which is nontrivial only for the sector $\mathcal{F}_N$, the proof remains the same. Denote by $\tilde{U}_N \subset z^{p_0}\mathbb{C}[z, z^{-1}]] \otimes \mathcal{F}$ the space

$$\tilde{U}_N = \pi_{N-1,i}^{-1}\left(\mathbb{C}[x_i] \otimes \Lambda^{(-)}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_N]\right).$$

We have the inclusion $\iota(\mathcal{F}_N) \subset \tilde{U}_N$. The analogous commutativity as in Lemma 2.3 holds:

$$\begin{array}{ccc} z^{p_0}\mathbb{C}[z, z^{-1}]] \otimes \mathcal{F}_{N-1} \supset \tilde{U}_N & \xrightarrow{\mathcal{A}} & \mathcal{F}_N \\ \downarrow{\scriptstyle \pi_{N-1,i}} & & \downarrow{\scriptstyle \pi_N} \\ \Lambda_-^{N,i} & \xrightarrow[\bar{\mathrm{A}}_N]{} & \Lambda_-^N \end{array}. \tag{2.34}$$

The proof may be reproduced as in Lemma 2.3 changing each occurrence of p_0 by $N-1$ due to $\tilde{U}_N \in z^{p_0}\mathbb{C}[z, z^{-1}]] \otimes \mathcal{F}_{N-1}$. Thus we have the commutative diagram for the Dunkl operators which is nontrivial for the N-th sector of the Fock space $\mathcal{F}_N$:

$$\begin{array}{ccc} \mathcal{F}_{N-1} \otimes \mathbb{C}[z, z^{-1}]] \supset \tilde{U}_N & \xrightarrow{\pi_{N-1,i}} & \Lambda_-^{N,i} \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle \bar{D}_i^{(N)}} \\ \mathcal{F}_{N-1} \otimes \mathbb{C}[z, z^{-1}]] \supset \tilde{U}_N & \xrightarrow[\pi_{N-1,i}]{} & \Lambda_-^{N,i} \end{array}. \tag{2.35}$$

On the other sectors of the Fock space (2.35) holds due to $\pi_{N-1,i}$ projects all to zero. We arrive to the following

Proposition 2.5 *The Hamiltonians $\mathscr{H}_k : \mathcal{F} \to \mathcal{F}$ are the pullback of Hamiltonians $\bar{H}_k^{(N)}$ with respect to the maps π_N.*

In other words, the Hamiltonians (2.31) obey the commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathscr{H}_k"] \arrow[d, "\pi_N"'] & \mathcal{F} \arrow[d, "\pi_N"] \\ \Lambda_-^N \arrow[r, "\bar{H}_k^{(N)}"'] & \Lambda_-^N \end{tikzcd}. \tag{2.36}$$

2. Now we want to describe the construction in the fermionic Fock space realized as semi-infinite wedges and present the projection analogous to π_N. We introduce the Clifford algebra generated by fermions ψ_k, ψ_k^* for $k \in \mathbb{Z}$ with anti-commutation relations

$$\psi_i\psi_j + \psi_j\psi_i = \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0,$$

$$\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}.$$

The fermionic Fock space $\mathcal{F}$ can be defined as a representation of the Clifford algebra, where the vacuum vetor $|0\rangle$ is defined as follows:

$$\psi_n|0\rangle = 0 \quad n \geq 0, \quad \psi_n^*|0\rangle = 0 \quad n < 0. \tag{2.37}$$

According to (2.37) the fermionic normal ordering $\vdots\,\vdots$ is defined as follows:

$$\vdots\psi_i^*\psi_j\vdots = \begin{cases} \psi_i^*\psi_j, & j \geq 0 \\ -\psi_j\psi_i^*, & j < 0 \end{cases}.$$

In other words all annihilation operators are moved to the right and all creation operators are moved to the left taking into account that the factor (-1) appears after exchanging neighboring fermionic operators. Any wedge in the space $\Lambda^{\frac{\infty}{2}}(\mathbb{C}[z, z^{-1}])$ can be obtained by acting of fermionic operators on the vacuum state

$$\vdots\psi_{k_1}\psi_{k_2}\ldots\psi_{k_n}\psi_{l_1}^*\psi_{l_2}^*\ldots\psi_{l_m}^*\vdots|0\rangle. \tag{2.38}$$

A charge of element (2.38) can be defined as $m - n$. We introduce the shifted vacuum $|c\rangle$

$$|c\rangle = \begin{cases} \psi_{c-1}^*\ldots\psi_1^*\psi_0^*|0\rangle & c > 0 \\ \psi_c\ldots\psi_{-2}\psi_{-1}|0\rangle & c < 0 \end{cases}.$$

In $\mathcal{F}$ we can choose a basis $|\lambda, c\rangle$ parameterized by partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$:

$$|\lambda, c\rangle = \psi_{\lambda_1-1}^*\psi_{\lambda_2-2}^*\ldots\psi_{\lambda_n-n}^*|c-n\rangle. \tag{2.39}$$

For fixed c vectors $|\lambda, c\rangle$ generate the c-th sector $\mathcal{F}_c$ of the fermionic Fock space as a vector space.

The fermionic Fock space admits a presentation $\mathcal{F} \cong \Lambda^{\frac{\infty}{2}}(\mathbb{C}[z, z^{-1}])$ in "semi-infinite wedges":

$$z^{k_1} \wedge z^{k_2} \wedge \ldots \wedge z^{k_m} \wedge \ldots, \quad k_1 > k_2 > \cdots > k_m > \ldots, \quad k_{n+1} = k_n - 1 \text{ all } n > N,$$

which form a basis of $\mathcal{F}$. The vacuum state $|0\rangle$ corresponds to

$$|0\rangle = z^{-1} \wedge z^{-2} \wedge z^{-3} \wedge z^{-4} \wedge \dots$$

An action of fermionic operators on the wedge v is presented by formulas:

$$\psi_n(v) = \frac{\partial}{\partial z^n} v, \quad \psi_n^*(v) = z^n \wedge v.$$

Note that the element z^n is added by ψ_n^* at the very beginning of the sequence, so the permutiaion with other elements may produce a sign. The symbol $\frac{\partial}{\partial z^n}$ means that if the wedge $v = z^n \wedge w$ then

$$\frac{\partial}{\partial z^n}(z^n \wedge w) = w.$$

The shifted vacuum is given by

$$|c\rangle = z^{c-1} \wedge z^{c-2} \wedge z^{c-3} \wedge z^{c-4} \dots.$$

and $|\lambda, c\rangle$ from (2.39)

$$|\lambda, c\rangle = z^{\lambda_1+c-1} \wedge z^{\lambda_2+c-2} \wedge \dots \wedge z^{\lambda_k+c-k} \wedge \dots \wedge z^{\lambda_n+c-n} \wedge z^{-n-1+c} \wedge z^{-n-2+c} \dots.$$

Define the space $\Lambda^N(\mathbb{C}[z, z^{-1}])$ of finite wedge $z_1^{k_1} \wedge z_2^{k_2} \wedge \dots \wedge z_N^{k_N}$ with N elements. It can be identified with the antisymmetric function $\Lambda^N(\mathbb{C}[z, z^{-1}]) \simeq \Lambda_-[z_1^{\pm 1}, \dots, z_N^{\pm 1}]$:

$$z_1^{k_1} \wedge z_2^{k_2} \wedge \dots \wedge z_N^{k_N} \Longleftrightarrow \mathrm{Alt}(z_1^{k_1}, \dots, z_N^{k_N}) = \det_{i,j=1\dots N} z_i^{k_j}. \tag{2.40}$$

For wedge $v = z^{k_1} \wedge z^{k_2} \wedge \dots \wedge z^{k_i} \wedge \dots \in \Lambda^{\frac{\infty}{2}}(\mathbb{C}[z, z^{-1}])$ denote by $p_0(v)$ the charge of v. We can define the embedding $\omega_N : \Lambda^{\frac{\infty}{2}}(\mathbb{C}[z, z^{-1}]) \to \Lambda^N(\mathbb{C}[z, z^{-1}])$:

$$\omega_N(v) = \begin{cases} z_1^{k_1} \wedge z_2^{k_2} \wedge \dots \wedge z_N^{k_N} & \text{if } p_0(v) = N \\ 0 & \text{if } p_0(v) \neq N \end{cases} \tag{2.41}$$

that simply keep only the first N elements in wedge v if its charge equals N. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ we have

$$\omega_N(|\lambda, c\rangle) = \begin{cases} z_1^{\lambda_1+N-1} \wedge z_2^{\lambda_2+N-2} \wedge \dots \wedge z_N^{\lambda_N} & \text{if } c = N \\ 0 & \text{if } c \neq N \end{cases},$$

where we put $\lambda_i = 0$ for $i > n$. Due to the isomorphism (2.40) we obtain

$$\omega_N(|\lambda, N\rangle) \simeq \det_{i,j=1\dots N} z_i^{\lambda_j+N-j} = \prod_{i<j} (z_i - z_j)\, s_\lambda(z_1, z_2, \dots, z_N),$$

where $s_\lambda(z_1, z_2, \dots, z_N)$ is a Schur polynomial. Define operators

$$a_n = \sum_j \vdots \psi_j^* \psi_{j+n} \vdots \tag{2.42}$$

It can be checked that they commute as bosonic operators

$$[a_k, a_l] = k\delta_{k+l,0}.$$

Define the operator Q with the following commutation relations

$$[a_n, Q] = \delta_{0,n}.$$

The operator e^Q is an operator which shifts the charge of the fermionic vector :

$$e^Q \psi_n e^{-Q} = \psi_{n+1}, \quad e^Q \psi_n^* e^{-Q} = \psi_{n+1}^*.$$

Define the fermion field $\Psi(x) = \sum_k \psi_k x^k$ and $\Psi^*(x) = \sum_k \psi_k^* x^{-k-1}$ with

$$\Psi^*(x)\Psi(x') = \frac{1}{x'-x} + \vdots\Psi^*(x)\Psi(x')\vdots \qquad x < x'. \tag{2.43}$$

The boson fermion correspondence is given by the formula (2.42) and the following relations:

$$\Psi(x) =: x^{a_0} e^Q \exp\left(-\sum_{n>0} \frac{a_{-n}}{nx^n}\right) \exp\left(\sum_{n>0} \frac{a_n}{n} x^n\right) : \tag{2.44}$$

$$\Psi^*(x) =: x^{-a_0} e^{-Q} \exp\left(\sum_{n>0} \frac{a_{-n}}{nx^n}\right) \exp\left(-\sum_{n>0} \frac{a_n}{n} x^n\right) :$$

This corresponds with the notations given at the beginning of this paragraph where we put:

$$a_{-n} = p_n, \quad a_n = n\frac{\partial}{\partial p_n} \text{ for } n > 0,$$

$$a_0 = p_0, \ Q = -\frac{\partial}{\partial p_0}.$$

and with notations of vertex operators (2.19) which are representation of $\psi(z)$ and $\psi^*(z)$. Due to the boson-fermion correspondence we formulate the following

Proposition 2.6 *The diagram (2.45) is commutative for $N > 0$.*

$$\begin{tikzcd} \mathcal{F}^{bos} \arrow[r, leftrightarrow] \arrow[d, "\pi_N"'] & \mathcal{F}^{fer} \arrow[d, "\omega_N"] \\ \Lambda_-^N \arrow[r, leftrightarrow] & \Lambda^N(\mathbb{C}[z, z^{-1}]) \end{tikzcd} \tag{2.45}$$

Here the upper isomorphism is the boson-fermion correspondence (2.42), (2.44). The lower isomorphism is given by (2.40).

Proof. Consider a vector $|\lambda, c\rangle \in \mathcal{F}_c^{fer}$ for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. We have shown that

$$\omega_N(|\lambda, c\rangle) = \begin{cases} \prod_{i<j}(z_i - z_j)\, s_\lambda(z_1, z_2, \ldots, z_N) & \text{if } c = N \\ 0 \ \text{ if } c \neq N \end{cases},$$

for $n \leq N$. One can show [3] that boson-fermion correspondence implies $|\lambda, c\rangle \simeq s_\lambda(\mathbf{p})|c\rangle$, where $s_\lambda(\mathbf{p})$ is a Schur polynomial in terms of p_k. Applying (2.33) to $s_\lambda(\mathbf{p})|c\rangle$ we obtain

$$\pi_N(s_\lambda(\mathbf{p})|c\rangle) = \begin{cases} \prod_{i<j}(z_i - z_j)\, s_\lambda(z_1, z_2, \ldots, z_N) & \text{if } c = N \\ 0 \ \text{ if } c \neq N \end{cases}.$$

■

3 Generating functions of commuting Hamiltonians for some special values of the coupling constant

In this section we derive the formulas for the densities of commuting Hamiltonians (2.31) with $\beta = 0$. In this case the Dunkl operator (2.29) is simply the differential operator $\left(z\frac{\partial}{\partial z}\right)$ and the Hamiltonians are given by

$$\mathscr{H}_k = \frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0} dz \int_{u\circlearrowleft z} du \frac{\Psi^*(u)}{u-z}\left(z\frac{\partial}{\partial z}\right)^k \Psi(z). \tag{3.1}$$

They are the pullbacks of (1.7) in case $\beta = 0$:

$$\bar{H}_k^{(N)} = \sum_{i=1}^{N}\left(x_i\frac{\partial}{\partial x_i}\right)^k.$$

In this section we assume that $\beta = 0$ and use the same notations for Hamiltonians as in (2.31). In this case the Hamiltonians can be simply expressed as operators on the fermionic Fock space

$$\mathscr{H}_n = \sum_k \vdots k^n \psi_k^* \psi_k \vdots.$$

The boson-fermion correspondence allows to express the Hamiltonians (3.1) in the bosonic Fock space, it was done by A. Pogrebkov [45] for the additive version and later by P. Rossi [50] on the circle. In other notations the correspondence for the exponential generating function of Hamiltonians is given by the formula [50]:

$$\mathscr{H}(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}\mathscr{H}_n = \frac{B(x)}{2\pi i x}\int\limits_{z\circlearrowleft 0}\frac{dz}{z}\left(: e^{xS(xz\frac{\partial}{\partial z})\varphi(z)} : -1\right). \tag{3.2}$$

Here

$$S(t) = \frac{\sinh(\frac{t}{2})}{\frac{t}{2}},$$

$$B(x) = \frac{x}{e^x - 1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n \tag{3.3}$$

— exponential generating function for Bernoulli numbers and a derivative of the bosonic field expressed in terms of in terms of (1.14)

$$\varphi(z) = \varphi^-(z) + \varphi^+(z). \tag{3.4}$$

In this section we provide the explicit expressions of the densities for the family of commuting Hamiltonians (3.1) different from (3.2). We deduce two different formulas. We derive first formula by calculating the integral in variable u in (3.1) using (2.19, 2.20), this leads to the following answer

$$\mathscr{H}(x) = \frac{B(x)}{2\pi i x}\int\limits_{z\circlearrowleft 0}\frac{dz}{z}\left(: \exp\left(x\left(z\frac{\partial}{\partial z} + \varphi(z)\right)\right) : -1\right), \tag{3.5}$$

where the exponent is a formal series and $B(x)$ is given by (3.3).

The second formula can be obtained by fermionic calculus. The answer will be given in terms of the integral operator $K : \mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]] \to \mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$

$$K\left[f(z)\right] = \frac{1}{2\pi i}\int\limits_{w\circlearrowleft z} \frac{dw}{w-z}\varphi(w)f(z),$$

$f(z)\in\mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$. Then the generating function can be expressed by the following formula

$$\mathscr{H}(x) = \frac{B(x)}{2\pi i x}\int\limits_{z\circlearrowleft 0} dz\frac{e^{xK}-1}{K}\left[\frac{\varphi(z)}{z}\right] \tag{3.6}$$

Here the operator $T(x,K) = \frac{e^{xK}-1}{K}$ is understood as a formal power series, see (3.22).

3.1 Bosonic calculations with vertex operators

Denote by $\mathscr{W}_k(z)$ the following integral

$$\mathscr{W}_k(z) = \frac{1}{2\pi i}\int_{u\circlearrowleft z} du\frac{\Psi^*(u)}{u-z}\left(z\frac{\partial}{\partial z}\right)^k\Psi(z) \tag{3.7}$$

and by $\mathscr{W}(z,x)$ the exponential generating function.

$$\mathscr{W}(z,x) = \sum_{n=0}^{\infty}\frac{\mathscr{W}_n(z)}{n!}x^n. \tag{3.8}$$

Then $\mathscr{W}_k(z)$ is the corresponding density of the Hamiltonian $\mathscr{H}_k$:

$$\mathscr{H}_k = \frac{1}{2\pi i}\int_{z\circlearrowleft 0}\frac{dz}{z}\mathscr{W}_k(z).$$

Densities $\mathscr{W}_k(z)$, as we shall see (3.20), can be written as fermionic normal ordered expression

$$\mathscr{W}_k(z) = \vdots\Psi^*(z)\left(z\frac{\partial}{\partial z}\right)^k\Psi(z)\vdots. \tag{3.9}$$

Our aim is to find the expressions for the densities $\mathscr{W}_k$ and the generating function (3.8) and to prove formula (3.5).

Denote by

$$\eta(z) = p_0\ln(z) - \sum_{n>0}\frac{p_n}{nz^n} + \sum_{n\geq 0} z^n\frac{\partial}{\partial p_n}$$

the bosonic field and, its derivative $\varphi(z) = \left(z\frac{\partial}{\partial z}\right)\eta(z) = \varphi^-(z)+\varphi^+(z)$ given in (1.14). Then we can also rewrite the operators (2.19), (2.20) in the following form

$$\Psi(z) =: e^{\eta(z)} : \qquad \Psi^*(z) =: e^{-\eta(z)} :, \tag{3.10}$$

where : : means bosonic normal ordering — all operators $\frac{\partial}{\partial p_n}$ are placed to the right and operators p_n are placed to the left. The simple relation follows

$$: \Psi^*(z)\Psi(z) := 1. \tag{3.11}$$

Denote by $w_n(z) =: \Psi^*(z)\left(z\frac{\partial}{\partial z}\right)^n \Psi(z) :$, it satisfies the recurrent relation:

$$w_{n+1}(z) =: \varphi(z)w_n(z) : +z\frac{\partial}{\partial z}w_n(z), \tag{3.12}$$

which follows from (3.10). In comparison with (3.9) $w_n(z)$ is boson normal ordered. The expressions for $n = 0, 1, 2, 3, 4$ are given in section 3.3. Combine $w_k(z)$ into a generating function $w(z, x)$:

$$w(z, x) = \sum_{k=0}^{\infty} w_k(z)\frac{x^k}{k!} =: \exp\left(x\left(z\frac{\partial}{\partial z} + \varphi(z)\right)\right) : . \tag{3.13}$$

Here the exponent of operator means the formal series

$$\exp\left(x\left(z\frac{\partial}{\partial z} + \varphi(z)\right)\right) = 1 + x\left(z\frac{\partial}{\partial z} + \varphi(z)\right) + \frac{x}{2!}\left(z\frac{\partial}{\partial z} + \varphi(z)\right)\left(z\frac{\partial}{\partial z} + \varphi(z)\right) + \ldots$$

acting on the constant function. Due to (3.12) we can write the differential equation on the generating function $w(z, x)$:

$$\frac{\partial w(z,x)}{\partial x} =: \varphi(z)w(z,x) : +z\frac{\partial w(z,x)}{\partial z}. \tag{3.14}$$

Futher we want to express $\mathscr{W}_k(z)$ from $w_n(z)$, we formulate the answer in Proposition 3.1. We devide the proof in three lemmas.

We use the following notations for $u_n(z) =: \Psi(z)\left(z\frac{\partial}{\partial z}\right)^n \Psi^*(z) :$ with the recurrent relations:

$$u_{n+1}(z) = - : \varphi(z)u_n(z) : +z\frac{\partial}{\partial z}u_n(z)$$

and generating function $u(z, x)$

$$u(z, x) = \sum_{k=0}^{\infty} u_k(z)\frac{x^k}{k!} =: \exp\left(x\left(z\frac{\partial}{\partial z} - \varphi(z)\right)\right) : .$$

Denote by

$$v_n(z) =: \Psi^*(z)z^n\frac{\partial^n}{\partial z^n}\Psi(z) : \quad q_n(z) =: \Psi(z)z^n\frac{\partial^n}{\partial z^n}\Psi^*(z) : . \tag{3.15}$$

We have the following expressions for their exponential generating functions:

$$v(z, x) = \sum_{k=0}^{\infty} v_k(z)\frac{x^k}{k!} =: \Psi^*(z)\Psi(z + xz) :,$$

$$q(z, x) = \sum_{k=0}^{\infty} q_k(z)\frac{x^k}{k!} =: \Psi(z)\Psi^*(z + xz) : .$$

Now we formulate a well known fact about exponential generating functions by the following

Lemma 3.1 *Let* $A(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$ *and* $B(x) = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}$ *be exponential generating function,* $C(x) = A(x)B(x)$ *is a product of* $A(x)$ *and* $B(x)$*. Then* $C(x) = \sum_{k=0}^{\infty} c_k \frac{x^k}{k!}$ *is an exponential generating function with coefficients* $c_n(x)$ *expressed by the following formula:*

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Now we formulate several technical lemmas:

Lemma 3.2 *The following relations are valid:*

(i) $: \sum_{k=0}^{l} \binom{l}{k} u_k(z) w_{l-k}(z) := \delta_{l,0}$,

(ii) $: \sum_{k=0}^{l} \binom{l}{k} v_k(z) q_{l-k}(z) := \delta_{l,0}$,

where $\delta_{l,0}$ *is the Kronecker delta.*

Proof. This is an immediate corollary of Lemma 3.1 and the relations for the generating functions:

$$: u(z,x)w(z,x) :=: \exp\left(x\left(z\frac{\partial}{\partial z} + \varphi(z)\right)\right) \exp\left(x\left(z\frac{\partial}{\partial z} - \varphi(z)\right)\right) := 1,$$

$$: v(z,x)q(z,x) :=: \Psi^*(z)\Psi(z+xz)\Psi(z)\Psi^*(z+xz) := 1.$$

■

Lemma 3.3 *The antisymmetrization* $\mathcal{A}$ *given in (2.26) of function* $z^n \frac{\partial^n}{\partial z^n}\Psi(z)$ *is given by the following formula:*

$$\mathcal{A}\left(z^n \frac{\partial^n}{\partial z^n}\Psi(z)\right) = \frac{1}{n+1}\left(\frac{1}{2\pi i}\right)\int_{z\circlearrowleft 0} \frac{dz}{z} : \Psi^*(z)\left(z^{n+1}\frac{\partial^{n+1}}{\partial z^{n+1}}\Psi(z)\right) : \tag{3.16}$$

Proof. We have to prove that

$$\frac{1}{2\pi i}\int_{y\circlearrowleft z} dy \frac{\Psi^*(y)}{y-z}\left(z^n \frac{\partial^n}{\partial z^n}\Psi(z)\right) = \frac{1}{n+1}\frac{v_{n+1}(z)}{z}.$$

Differentiating n times (2.21) by z and multiplying by z^n we obtain:

$$\Psi^*(y)\left(z^n\frac{\partial^n}{\partial z^n}\right)\Psi(z) = \sum_{k=0}^{n}\binom{n}{k}\frac{k!\,(-z)^k}{(z-y)^{k+1}} : \left(z^{n-k}\frac{\partial^{n-k}}{\partial z^{n-k}}\right)\Psi(z)\Psi^*(y) : .$$

After the normal ordering we calculate the integral:

$$\frac{1}{2\pi i}\int_{y\circlearrowleft z} dy \frac{\Psi^*(y)}{y-z}\left(z^n \frac{\partial^n}{\partial z^n}\Psi(z)\right) =$$

$$= -\frac{1}{z}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2\pi i}\int_{y\circlearrowleft z} dy \frac{k!\,z^{k+1}}{(y-z)^{k+2}} : \left(z^{n-k}\frac{\partial^{n-k}}{\partial z^{n-k}}\right)\Psi(z)\Psi^*(y) :=$$

$$= -\frac{1}{z}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1} : q_{k+1}(z)v_{n-k}(z) := -\frac{1}{z(n+1)}\sum_{k=0}^{n}\binom{n+1}{k+1} : q_{k+1}(z)v_{n-k}(z) := \frac{v_{n+1}(z)}{z(n+1)}.$$

In the last equality we use Lemma 3.2. ∎

We use the following notations for coefficients which connect the expression for two types of derivatives $\left(z\frac{\partial}{\partial z}\right)^n$ and $z^n\frac{\partial^n}{\partial z^n}$:

$$\left(z\frac{\partial}{\partial z}\right)^n = \sum_{k=1}^{n} a_{n,k} z^k \frac{\partial^k}{\partial z^k},$$

$$z^n\frac{\partial^n}{\partial z^n} = \sum_{k=1}^{n} b_{n,k}\left(z\frac{\partial}{\partial z}\right)^k.$$

One can check the recurrent relations:

$$a_{n,k} = k a_{n-1,k} + a_{n-1,k-1}, \quad a_{0,n} = \delta_{0,n},$$

$$b_{n,k} = -(n-1)b_{n-1,k} + b_{n-1,k-1}, \quad b_{0,n} = \delta_{0,n}.$$

Denote by B_n the Bernoulli number given by the exponential generating function (3.3) or equivalently by the recurrent formula:

$$B_0 = 1,$$

$$B_n = \frac{-1}{n+1}\sum_{k=1}^{n}\binom{n+1}{k+1}B_{n-k}.$$

The following lemma can be proved by the induction

Lemma 3.4 *The following relation is valid:*

$$\sum_{l=m}^{n}\frac{1}{l}a_{n-1,l-1}b_{l,m} = \frac{1}{n}B_{n-m}\binom{n}{m}.$$

Now we can provide an expression for the density (3.7) and the generating function (3.8). We formulate the following

Proposition 3.1 *The density $\mathscr{W}_n(z)$ is expressed by the formula:*

$$\mathscr{W}_{n-1}(z) = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}B_{n-k}w_k(z), \tag{3.17}$$

the exponential generating function (3.8) is given by the formula

$$\mathscr{W}(z,x) = \frac{:\exp\left(x\left(z\frac{\partial}{\partial z} + \varphi(z)\right)\right): -1}{e^x - 1}. \tag{3.18}$$

and satisfies the differential equation

$$\frac{\partial \mathscr{W}(x,z)}{\partial x} =: \varphi(z)\mathscr{W}(z,x) : + z\frac{\partial \mathscr{W}(x,z)}{\partial z} - \frac{e^x\mathscr{W}(z,x) - \varphi(z)}{e^x - 1}. \tag{3.19}$$

Proof. By definition

$$\mathscr{W}_{n-1}(z) = \frac{z}{2\pi i}\int_{y\circlearrowleft z} dy \frac{\Psi^*(y)}{y-z}\left(z\frac{\partial}{\partial z}\right)^{n-1}\Psi(z) = \frac{z}{2\pi i}\int_{y\circlearrowleft z} dy \frac{\Psi^*(y)}{y-z}\sum_{k=1}^{n-1} a_{n-1,k}\left(z^k\frac{\partial^k}{\partial z^k}\Psi(z)\right).$$

Using Lemma 3.3 we obtain:

$$\begin{aligned}\mathscr{W}_{n-1}(z) &=: \Psi^*(z)\left(\sum_{k=1}^{n-1}\frac{1}{k+1}a_{n-1,k}z^{k+1}\frac{\partial^{k+1}}{\partial z^{k+1}}\Psi(z)\right): \\ &=: \Psi^*(z)\sum_{k=1}^{n-1}\frac{1}{k+1}a_{n-1,k}\sum_{m=1}^{k+1} b_{k+1,m}\left(z\frac{\partial}{\partial z}\right)^m\Psi(z): \\ &= \sum_{m=1}^{n}\sum_{k=m-1}^{n-1}\frac{1}{k+1}a_{n-1,k}b_{k+1,m}w_m(z) = \frac{1}{n}\sum_{m=1}^{n}\binom{n}{m}B_{n-m}w_m(z)\end{aligned}$$

In the last equality we use Lemma 3.4. Thus we obtain (3.17).

Now we check the formula for the generating function. We have two exponential generating functions:

$$\frac{1}{x}\left(w(z,x)-1\right) = \frac{1}{x}\left(\sum_{k=1}^{\infty} w_k(z)\frac{x^k}{k!}\right) = \sum_{k=0}^{\infty}\frac{w_{k+1}(z)}{k+1}\frac{x^k}{k!}$$

and $B(x)$ given by (3.3). Using Statement 3.1 we check that

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n-1}{k}B_{n-1-k}w_{k+1}(z) &= \sum_{m=1}^{n}\frac{1}{m}\binom{n-1}{m-1}B_{n-m}w_m(z) \\ &= \frac{1}{n}\sum_{m=1}^{n}\binom{n}{m}B_{n-m}w_m(z) = \mathscr{W}_{n-1}(z).\end{aligned}$$

The differntial equation (3.19) directly follows from (3.14). ■

We proved that the densities $\mathscr{W}_k(z)$ are linearly expressed by $w_n(z)$ (3.13) and present these expressions for the first densities:

$$\mathscr{W}_0(z) = w_1(z), \qquad \mathscr{W}_1(z) = \frac{1}{2}w_2(z) - \frac{1}{2}w_1(z),$$

$$\begin{aligned}\mathscr{W}_2(z) &= \frac{1}{3}w_3(z) - \frac{1}{2}w_2(z) + \frac{1}{6}w_1(z), \\ \mathscr{W}_3(z) &= \frac{1}{4}w_4(z) - \frac{1}{2}w_3(z) + \frac{1}{4}w_2(z).\end{aligned}$$

The expressions for the corresponding Hamiltonians are given below

$$\mathscr{H}_0 = p_0, \qquad \mathscr{H}_1 = \sum_{n>0} np_n\frac{\partial}{\partial p_n} + \frac{1}{2}\left(p_0^2 - p_0\right),$$

$$\begin{aligned}\mathscr{H}_2 = \sum_{n,k>0} nkp_{n+k}\frac{\partial}{\partial p_n}\frac{\partial}{\partial p_k} + \sum_{n,k>0}(n+k)p_np_k\frac{\partial}{\partial p_{n+k}} + \\ +(2p_0-1)\sum_{n>0} np_n\frac{\partial}{\partial p_n} + \frac{1}{6}(2p_0^3 - 3p_0^2 + p_0),\end{aligned}$$

$$
\begin{aligned}
\mathscr{H}_3 = & \sum_{n,k,m>0} nkmp_{n+k+m}\frac{\partial}{\partial p_n}\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_m} + \sum_{n,k,m>0}(n+k+m)p_np_kp_m\frac{\partial}{\partial p_{n+k+m}} + \\
& + \frac{3}{2}\sum_{k,m>0}\sum_{n=1}^{m+k-1} kmp_np_{m+k-n}\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_m} + \frac{1}{2}\sum_{n>0} n^3 p_n\frac{\partial}{\partial p_n} + \\
& + \left(3p_0 - \frac{3}{2}\right)\sum_{n,k>0} nkp_{n+k}\frac{\partial}{\partial p_n}\frac{\partial}{\partial p_k} + \left(3p_0-\frac{3}{2}\right)\sum_{n,k>0}(n+k)p_np_k\frac{\partial}{\partial p_{n+k}} \\
& + \left(3p_0^2 - 3p_0 + \frac{1}{2}\right)\sum_{n>0} np_n\frac{\partial}{\partial p_n} + \frac{1}{4}\left(p_0^4 - 2p_0^3 + p_0^2\right).
\end{aligned}
$$

3.2 Boson-fermion correspondence and integral operators

Due to this correspondence we can compute the Hamiltonians (3.1) in fermionic variables using (2.43):

$$
\mathscr{H}_k = \frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0} dz \int_{u\circlearrowleft z} du \frac{\Psi^*(u)}{u-z}\left(z\frac{\partial}{\partial z}\right)^k \Psi(z) = \frac{1}{2\pi i}\int_{z\circlearrowleft 0} dz \vdots\Psi^*(z)\left(z\frac{\partial}{\partial z}\right)^k\Psi(z)\vdots, \tag{3.20}
$$

which gives expressions (**??**) for the Hamiltonians in the fermionic Fock space.

The Hamiltonians in the bosonic Fock space one can obtain directly form the boson-fermion correspondence :

$$
\vdots\Psi^*(x)\Psi(x')\vdots = \Psi^*(x)\Psi(x') - \frac{1}{x-x'} \to \frac{:\Psi^*(x)\Psi(x') - 1:}{x-x'}.
$$

Putting $x' = x - \varepsilon x$ and expending into the series by ε one can see the densities for the linear combinations of Hamiltonians $\mathscr{H}_k$. The answer is given by formula (3.2) in [45, 50].

Another way to express the answer is to use the fermionic calculus. Due to (2.42) we have

$$
\varphi(z) = \vdots z\Psi^*(z)\Psi(z)\vdots
$$

Let $\mathscr{U}_n(z)$ be the n-th density of Hamiltonian

$$
\mathscr{H}_n = \frac{1}{2\pi i}\int\limits_{z\circlearrowleft 0}\mathscr{U}_n(z)dz = \frac{1}{2\pi i}\int\limits_{z\circlearrowleft 0}\vdots\Psi^*(z)z\frac{\partial}{\partial z}\Psi(z)\vdots.
$$

We start with $\mathscr{U}_0(z) = \frac{\varphi(z)}{z}$. To calculate the first density we use Wick's theorem

$$
\begin{aligned}
\varphi(z)\mathscr{U}_0(w) = & \vdots z\Psi^*(z)\Psi(z)\vdots\vdots\Psi^*(w)\Psi(w)\vdots = \\
& = \vdots z\Psi^*(z)\Psi(z)\Psi^*(w)\Psi(w)\vdots + \frac{z}{w-z}\vdots\Psi^*(z)\Psi(w)\vdots \\
& \qquad + \frac{z}{w-z}\vdots\Psi(z)\Psi^*(w)\vdots + \frac{z^2}{(w-z)^2}
\end{aligned}
$$

Computing the integral

$$
\frac{1}{2\pi i}\int\limits_{w\circlearrowleft z}\frac{dw}{w-z}\varphi(z)\mathscr{U}_0(w) = \vdots\Psi^*(z)z\frac{\partial}{\partial z}\Psi(z)\vdots + \vdots\Psi(z)z\frac{\partial}{\partial z}\Psi^*(z)\vdots
$$

we obtain

$$\vdots\Psi^*(z)z\frac{\partial}{\partial z}\Psi(z)\vdots = \frac{1}{2}\left(\frac{1}{(2\pi i)}\int\limits_{w\circlearrowleft z}\frac{dw}{w-z}\varphi(z)\mathscr{U}_0(w) + z\frac{\partial}{\partial z}\mathscr{U}_0(z)\right).$$

Using the integration by part we can write the following formula for the first density

$$\mathscr{U}(z)_1 = \frac{1}{2}\left(\frac{1}{2\pi i}\int\limits_{w\circlearrowleft z}\frac{dw}{w-z}\varphi(z)\mathscr{U}_0(w) - \mathscr{U}_0(z)\vdots\right)$$

In general case

$$\begin{aligned}\varphi(z)\mathscr{U}_k(w) =& \vdots z\Psi^*(z)\Psi(z)\vdots\vdots\Psi^*(w)\left(w\frac{\partial}{\partial w}\right)^k\Psi(w)\vdots = \\ =& \vdots z\Psi^*(z)\Psi(z)\Psi^*(w)\left(w\frac{\partial}{\partial w}\right)^k\Psi(w)\vdots + \frac{z}{w-z}\vdots\Psi^*(z)\left(w\frac{\partial}{\partial w}\right)^k\Psi(w)\vdots \\ &+ \left(\left(w\frac{\partial}{\partial w}\right)^k\frac{z}{w-z}\right)\vdots\Psi(z)\Psi^*(w)\vdots + \frac{z}{(w-z)}\left(\left(w\frac{\partial}{\partial w}\right)^k\frac{z}{w-z}\right).\end{aligned}$$

Using integration by parts one can express linearly $\mathscr{H}_{k+1}$ through the previous $\mathscr{H}_n$ and $\frac{1}{2\pi i}\int\limits_{w\circlearrowleft z}\frac{dw}{w-z}\varphi(z)\mathscr{U}_k(w)$. We omit the details, the calculations use the same combinatorial relation as in Lemma 3.4. To express the answer we introduce an integral operator $K : \mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]] \to \mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$

$$K[f(z)] = \frac{1}{2\pi i}\int\limits_{w\circlearrowleft z}\frac{dw}{w-z}\varphi(z)f(w),$$

with the kernel $K(z,w) = \dfrac{\varphi(z)}{w-z}$, $f(z)\in\mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$. Now we formulate that $\mathscr{U}_k$ can be obtained by consistent applications of operator K by the following

Proposition 3.2 *The Hamiltonians can be expressed by the formula:*

$$\mathscr{H}_n = \frac{1}{2\pi i}\int\limits_{z\circlearrowleft 0}dz\left(\frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l+1}B_{n-l}K^l\left[\frac{\varphi(z)}{z}\right]\right).$$

The generating function for the Hamiltonians is given by

$$\mathscr{H}(x) = \frac{B(x)}{2\pi i x}\int\limits_{z\circlearrowleft 0}dz\frac{e^{xK}-1}{K}\left[\frac{\varphi(z)}{z}\right]. \tag{3.21}$$

Here the operator $T(x,K) = \frac{e^{xK}-1}{K}$ means a formal power series in K:

$$\begin{aligned}T(x,K) &= xT_1 + x^2T_2 + x^3T_3 + \dots \\ &= x + \frac{x^2}{2}K + \frac{x^3}{6}K^2 + \frac{x^4}{24}K^3 + \dots,\end{aligned} \tag{3.22}$$

where $T_n = \frac{1}{n!}K^{n-1}$. Note that the expression (3.21) is not normal ordered. As an example we give the expressions for the first generators of $T(x,K)\left[\frac{\varphi(z)}{z}\right]$:

$$T_1\left[\frac{\varphi(z)}{z}\right] = \frac{\varphi(z)}{z}$$

$$T_2\left[\frac{\varphi(z)}{z}\right] = \frac{1}{2\pi i}\int\limits_{w\circlearrowleft z}\frac{dw}{w-z}\varphi(z)\frac{\varphi(w)}{w}$$

$$T_3\left[\frac{\varphi(z)}{z}\right] = \frac{1}{(2\pi i)^2}\int\limits_{u\circlearrowleft z}\frac{du}{u-z}\varphi(z)\int\limits_{w\circlearrowleft u}\frac{dw}{w-u}\varphi(u)\frac{\varphi(w)}{w}$$

3.3 Comparison of three constructions

Here we compare three formulas (3.5), (3.2) and (3.21) for the generating functions of the Hamiltonians. All the functions have the same multiplier $B(x)$ which produces the Bernoulli numbers. So the three ways to express the Hamiltonians is

$$\mathscr{H}(x) = \frac{B(x)}{2\pi i x}\int\limits_{z\circlearrowleft 0}\frac{dz}{z}\left(w(z,x)-1\right) = \frac{B(x)}{2\pi i x}\int\limits_{z\circlearrowleft 0}\frac{dz}{z}\left(s(z,x)-1\right) = \frac{B(x)}{2\pi i x}\int\limits_{z\circlearrowleft 0}dz\,T(x,K)\left[\frac{\varphi(z)}{z}\right]$$

where $w(z,x)$ is given by (3.13), $T(x,K)$ is given by (3.22) and

$$s(z,x) = 1 + \sum_{k>0}\frac{x^k}{k!}s_k(z)$$

is the exponential generating function given by the formula

$$s(z,x) =: e^{xS(xz\frac{\partial}{\partial z})\varphi(z)} : \quad \text{where} \qquad S(t) = \frac{\sinh(\frac{t}{2})}{\frac{t}{2}}.$$

The generating functions $s(z,x)$ and $w(z,x)$ differ by full derivative, we demonstrate this for the first generators:

$s_1(z) =: \varphi(z):$	$w_1(z) =: \varphi(z):$
$s_2(z) =: \varphi(z)^2:$	$w_2(z) =: (\varphi(z)^2 + \varphi'(z)):$
$s_3(z) =: (\varphi(z)^3 + \frac{1}{4}\varphi''(z)):$	$w_3(z) =: (\varphi(z)^3 + 3\varphi(z)\varphi'(z) + \varphi''(z)):$
$s_4(z) =: (\varphi(z)^4 + \varphi(z)\varphi''(z)):$	$w_4(z) =: (\varphi(z)^4 + 6\varphi^2(z)\varphi'(z):$ $+4\varphi(z)\varphi''(z) + 3(\varphi'(z)^2) + \varphi'''(z))$

Here $f'(z)$ means $z\frac{\partial}{\partial z}f(z)$. The densities $s_n(z)$ have less items, $w_n(z)$ have simple recurrent relations (3.12). In fact formula $w(z,x)$ gives precise expression for densities (3.7). The last formula (3.21) does not give a normal ordering answer, but is expressed as consistent application of simple integral operator in comparison with $w(z,x)$ and $s(z,x)$ which are expressed in terms of differential operators.

3.4 Time evolutions hierarchy

The Hamiltonians $\mathscr{H}_n$ commute, thus we can define an hierarchy of time evolutions defined these commutative flows as

$$\varphi_{t_n}(z) = [\mathscr{H}_n, \varphi(z)].$$

In fact it can be calculated directly and we formulate the result by the following

Lemma 3.5 *The hierarchy of time evolutions defined by commutative family (3.2) is given by*

$$\varphi_{t_k}(z) = \frac{1}{2}B(x) : \int_{x\circlearrowleft 0} dx \frac{k!}{x^{k+1}} \sinh\left(xz\frac{\partial}{\partial z}\right) e^{xS(xz\frac{\partial}{\partial z})\varphi(z)} : . \tag{3.23}$$

Proof. Due to commutator relations

$$\left[n\frac{\partial}{\partial p_n}, p_k\right] = n\delta_{n,k},$$

we have

$$[\varphi(\xi), \varphi(\eta)] = \xi\frac{\partial}{\partial \xi}\delta\left(\xi/\eta\right), \tag{3.24}$$

where $\delta\left(\xi/\eta\right) = \sum_n \frac{\xi^n}{\eta^n}$ means delta-function on the circle. Using (3.24) we compute the commutator relations with $s(y,x) =: e^{xS(xy\frac{\partial}{\partial y})\varphi(y)}$:

$$[\varphi(z), \int_{y\circlearrowleft 0} \frac{dy}{y}\sigma(y,x)] =: \frac{x}{2}\sinh\left(xz\frac{\partial}{\partial z}\right) e^{xS(xz\frac{\partial}{\partial z})\varphi(z)} :, \tag{3.25}$$

for the first $s_n(z)$ this gives:

$$[\varphi(z), \int_{y\circlearrowleft 0} \frac{dy}{y}s_1(y)] = 0, \qquad [\varphi(z), \int_{y\circlearrowleft 0} \frac{dy}{y}s_2(y)] = z\frac{\partial}{\partial z}\varphi(z)$$

$$[\varphi(z), \int_{y\circlearrowleft 0} \frac{dy}{y}s_2(y)] = z\frac{\partial}{\partial z}\left(: \varphi^2(z) :\right)$$

$$[\varphi(z), \int_{y\circlearrowleft 0} \frac{dy}{y}s_3(y)] = z\frac{\partial}{\partial z}\left(: \varphi^3(z) : + \frac{1}{2}\left(z\frac{\partial}{\partial z}\right)^2\varphi(z)\right)$$

$$[\varphi(z), \int_{y\circlearrowleft 0} \frac{dy}{y}s_4(y)] = z\frac{\partial}{\partial z}\left(: \varphi^4(z) : +2 : \varphi(z)\left(z\frac{\partial}{\partial z}\right)^2\varphi(z) : + : \left(z\frac{\partial}{\partial z}\varphi(z)\right)^2 :\right)$$

Up to the replacement the derivative $z\frac{\partial}{\partial z}$ by $\frac{\partial}{\partial z}$ the evolutions given by $[\varphi(z), \int_{y\circlearrowleft 0}\frac{dy}{y}s_k(y)]$ coincide with the hierarchy found in [45] for the additive version.

Using (3.25) and taking into account the Bernoulli factor $\frac{1}{e^x-1}$ we obtain the general formula (3.23). ∎

The first examples of (3.23) are given by:

$$\varphi_{t_0}(z) = 0, \quad \varphi_{t_1}(z) = \frac{1}{2}\varphi'(z), \qquad \varphi_{t_2}(z) =: z\frac{\partial}{\partial z}\left(\frac{1}{3}\varphi^2(z) - \frac{1}{2}\varphi(z)\right) :$$

$$\varphi_{t_3}(z) =: z\frac{\partial}{\partial z}\left(\frac{1}{4}\varphi^3(z) + \frac{1}{8}\left(z\frac{\partial}{\partial z}\right)^2\varphi(z) - \frac{1}{2}\varphi^2(z) + \frac{1}{4}\varphi(z)\right) :,$$

where $f'(z)$ means $z\frac{\partial}{\partial z}f(z)$. The classical limit of the hierarchy (3.23) is the dispersionless KdV hierarchy on the circle, see [45].

3.5 Generating functions for $\alpha = 0$

To find the generating function for higher Hamiltonians in general case is an open problem. We are able to calculate the Hamiltonians in case $\alpha = \frac{1}{\beta}$. Denote by $D' = \lim_{\alpha \to 0} \alpha D$, where D is given by (2.29) and $\alpha = \frac{1}{\beta}$, then

$$D'F(z) = \frac{1}{(2\pi i)^2} \int_{w \circlearrowleft 0} dw \int_{u \circlearrowleft w} \frac{du}{u-w} \frac{\Psi^*(u)}{1-\frac{w}{z}} \left(\Psi(w)F(z) - \Psi(z)F(w)\right).$$

Proposition 2.4 works for D': the operators $\mathscr{H}_k = (\iota(D')^k \mathcal{A})$ commute. Denote by $\mathscr{W}_k^-(z)$ the corresponding density and combine them into generating function:

$$\mathscr{W}^-(z,x) = \sum_{k=0}^{\infty} \frac{\mathscr{W}_k^-(z)}{k!} x^k.$$

Proposition 3.3 *The generating function is expressed by the formula:*

$$\mathscr{W}^-(z,x) = e^{-x}\left(1 + \frac{\partial}{\partial x}\left(\exp\left(xz\frac{\partial}{\partial z} + x\varphi^-(z)\right)\right)\right).$$

Here $\varphi^-(z)$ is given (1.14) and exponent means a formal series

$$\exp\left(xz\frac{\partial}{\partial z} + x\varphi^-(z)\right) = 1 + x\varphi^-(z) + \frac{x^2}{2}\left((\varphi^-(z))^2 + z\frac{\partial}{\partial z}\varphi^-(z)\right) + \dots$$

In fact this case gives a trivial answer. The densities $\mathscr{W}_k^-(z)$ depends on $\varphi^-(z)$ and its derivatives, so the integrals $\mathscr{H}_k = \frac{1}{2\pi i}\int \frac{dz}{z}\mathscr{W}_k^-(z)$ are polynomials in p_0 in this case.

4 Dunkl operators and representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_s)$

1. The Yangian $\mathrm{Y}(\mathfrak{gl}_s)$. Let

$$R_{12}(u-v) = 1 - \frac{1}{u-v}\sum_{a,b} E_{ab}\otimes E_{ba} \in \mathrm{End}\,(\mathbb{C}^s\otimes\mathbb{C}^s),$$

where $E_{ab}\in\mathrm{End}\,\mathbb{C}^s$ is the matrix unit, $E_{ab}(e^c)=\delta_{bc}e^a$ for basic vectors $e^c\in\mathbb{C}^s$. By definition, the Yangian $\mathrm{Y}(\mathfrak{gl}_s)$ is a unital associative algebra over $\mathbb{C}$ with generators $t_{ab,i}$, $a,b=1,...,s$, $i=0,1,...$ subject to the relations encoded in the Yang-Baxter equation

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v), \tag{4.1}$$

where $T_1(u)=T(u)\otimes\mathrm{Id}$, $T_2(u)=\mathrm{Id}\otimes T(u)$,

$$T(u)=\sum_{a,b=1}^{s} E_{ab}\otimes t_{ab}(u)\in\mathrm{End}(\mathbb{C}^s)\otimes\mathrm{Y}(\mathfrak{gl}_s)[u^{-1}]$$

and

$$t_{ab}(u)=\delta^{ab}+\sum_{i=0}^{\infty} t_{ab,i}u^{-i-1},\quad a,b=1,\dots s \tag{4.2}$$

are generating functions of $t_{ab,i}$. Equivalently, the defining relations (4.1) of $\mathrm{Y}(\mathfrak{gl}_s)$ are [36]

$$[t_{ab}(u),t_{cd}(v)] = \frac{t_{cb}(u)t_{ad}(v)-t_{cb}(v)t_{ad}(u)}{u-v}. \tag{4.3}$$

The center of the Yangian is generated by coefficients of the quantum determinant [9, 36].:

$$q\det t(u)=\sum_{\sigma\in S_m}(-1)^{sgn(\sigma)}t_{\sigma(1),1}(u)t_{\sigma(2),2}(u-1)...t_{\sigma(m),m}(u-m+1). \tag{4.4}$$

2. Representation of the degenerate affine Hecke algebra. Consider a space $V^{\otimes N}$ of functions in N variables with values in vector space $(\mathbb{C}^s)^{\otimes N}$, here[2] $V=\mathbb{C}[z]\otimes\mathbb{C}^s$. Denote by K_{ij} the coordinate exchange operator of i-th and j-th variable and by P_{ij} the permutation of i-th and j-th tensor copy of the vector space $\mathbb{C}^s$. We fix a basis $\{e^1,e^2,\dots,e^s\}$ in the $\mathbb{C}^s$ that we call spin space. The operators K_{ij} of permutation of the coordinates, P_{ij} of permutation of the spins, and $\sigma_{ij}=K_{ij}P_{ij}$ of the corresponding total action of the symmetric group S_N can be expressed by the following formulas

$$K_{ij}:\big(\ldots\otimes(e^{a_i}\otimes z^{k_i})\otimes\ldots\otimes(e^{a_j}\otimes z^{k_j})\otimes\ldots\big)\to\big(\ldots\otimes(e^{a_i}\otimes z^{k_j})\otimes\ldots\otimes(e^{a_j}\otimes z^{k_i})\otimes\ldots\big)$$

$$P_{ij}:\big(\ldots\otimes(e^{a_i}\otimes z^{k_i})\otimes\ldots\otimes(e^{a_j}\otimes z^{k_j})\otimes\ldots\big)\to\big(\ldots\otimes(e^{a_j}\otimes z^{k_i})\otimes\ldots\otimes(e^{a_i}\otimes z^{k_j})\otimes\ldots\big)$$

$$\sigma_{ij}:\big(\ldots\otimes(e^{a_i}\otimes z^{k_i})\otimes\ldots\otimes(e^{a_j}\otimes z^{k_j})\otimes\ldots\big)\to\big(\ldots\otimes(e^{a_j}\otimes z^{k_j})\otimes\ldots\otimes(e^{a_i}\otimes z^{k_i})\otimes\ldots\big)$$

Denote by $\Lambda_{\pm}^{s,N}$ the spaces of total invariants or respectively skewinvariants of the symmetric group S_N in the space $V^{\otimes N}$,

$$\Lambda_{\pm}^{s,N}=\left(V^{\otimes N}\right)^{(\pm)}. \tag{4.5}$$

[2] We also use the notation $V=V(z)$ when we need to specify the name of the variable

The (skew)invariants are taken with respect to the diagonal action of the symmetric groups, $\sigma_{ij} \mapsto K_{ij}P_{ij}$,

Next we describe the representation of the degenerate affine Hecke algebra in the space $\Lambda^{s,N}_{\pm}$. We use the Heckman–Dunkl operators $\mathcal{D}_i^{(N)} : V\otimes\Lambda^{s,N-1}_{\pm} \to V\otimes\Lambda^{s,N-1}_{\pm}$ in the form suggested by Polychronakos [46]:

$$\mathcal{D}_i^{(N)} = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{j\neq i} \frac{x_i}{x_i - x_j}\,(1-K_{ij})\,. \tag{4.6}$$

These operators do not change spins and satisfy the relations

$$K_{ij}\mathcal{D}_i^{(N)} = \mathcal{D}_j^{(N)}K_{ij},$$
$$[\mathcal{D}_i^{(N)},\mathcal{D}_j^{(N)}] = (\mathcal{D}_j^{(N)} - \mathcal{D}_i^{(N)})K_{ij},$$

which coincide with the relations of the degenerate affine Hecke algebra H_N.

There are two natural commuting families in degenerated affine Hecke algebra H_N, see [11]. The first one is formed by commuting operators ε_i [3]

$$\varepsilon_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{j<i} \frac{x_j}{x_i - x_j}\,(1-K_{ij}) + \sum_{i<j} \frac{x_i}{x_i - x_j}\,(1-K_{ij}) + (i-1)\,.$$

They commute, and satisfy the relations

$$K_{i,i+1}\varepsilon_i = \varepsilon_{i+1}K_{i,i+1} - 1. \tag{4.7}$$

Another family is formed by the elements

$$d_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{j<i} \frac{x_i}{x_i - x_j}\,(1-K_{ij}) + \sum_{i<j} \frac{x_j}{x_i - x_j}\,(1-K_{ij}) + (N-i)\,. \tag{4.8}$$

The elements d_i satisfy relations

$$K_{i,i+1}d_i = d_{i+1}K_{i,i+1} + 1 \tag{4.9}$$

and are related to ε_i as

$$d_i = K_0\varepsilon_{N-i}K_0, \tag{4.10}$$

where $K_0(x_i) = x_{N-i}$ represents the permutation of coordinates, associated to the longest element of the symmetric group. Heckman operators $\mathcal{D}_i$, see (4.6), which we use, are related to the above families by the relations

$$\mathcal{D}_i = K_{1i}\varepsilon_1K_{1i} = K_{iN}d_NK_{iN}, \qquad \mathcal{D}_i = \varepsilon_i - \sum_{j<i}K_{ij} = d_i - \sum_{j>i}K_{ij}. \tag{4.11}$$

3. Representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_s)$

Let

$$T(u) = \sum_{a,b=1}^{s} E_{ab}\otimes t_{ab}(u) \in \mathrm{End}(\mathbb{C}^s)\otimes \mathrm{Y}(\mathfrak{gl}_s)[u^{-1}]$$

[3]here and further we omit upper index in $\varepsilon_i^{(N)}$ and in $d_i^{(N)}$ assuming their dependence of N variables.

be the generating matrix of Yangian generators. Then the prescription

$$\xi_a : T(u) \to 1 + \frac{I^{(01)}}{u-a} \tag{4.12}$$

describes the evaluation homomorphism $\xi_a : \mathrm{Y}(\mathfrak{gl}_s) \to \mathrm{U}(\mathfrak{gl}_s)$. Here $I^{(01)} = \sum_{ab=1}^{s} E_{ab}^{(0)} \otimes E_{ab}^{(1)}$. The upper index in (4.12) specifies the tensor component. Since Yangian is the Hopf algebra and the operators $\varepsilon_k : \mathbb{C}[z]^{\otimes N} \to \mathbb{C}[z]^{\otimes N}$ commute, the assignement

$$T(u) \mapsto T_N(u) = \left(1 + \frac{I^{(01)}}{u \pm \varepsilon_1}\right) \cdots \left(1 + \frac{I^{(0n)}}{u \pm \varepsilon_N}\right) \tag{4.13}$$

determines a representation of $\mathrm{Y}(\mathfrak{gl}_s)$ in $V^{\otimes N} \simeq (\mathbb{C}^s)^{\otimes N} \otimes \mathbb{C}[z_1, ..., z_N]$. It is known [14, 5], that this Yangian action preserves the subspace $U_{s,N}^{\pm} = \sum_{i=1}^{N-1}(s_{i,i+1} \mp 1)V^{\otimes N}$ and thus equips the space

$$\tilde{\Lambda}_{\pm}^{s,N} = V^{\otimes N}/U_{s,N}^{\pm}$$

of S_N – (skew)coinvariants of $V^{\otimes N}$ with the structure of $\mathrm{Y}(\mathfrak{gl}_s)$ - module. Conjugation of the RHS of 4.13 by means the longest element $w_0 = K_0 P_0$ of the symmetric group S_N gives another presentation of the Yangian action in $\tilde{\Lambda}_{\pm}^{s,N}$:

$$T_N(u) = \left(1 + \frac{I^{(0N)}}{u \pm d_N}\right) \cdots \left(1 + \frac{I^{(01)}}{u \pm d_1}\right). \tag{4.14}$$

T.Arakawa proved [5, Proposition 5], that in antisymmetric case

$$T_N(u) \equiv 1 + \sum_{i=1}^{N} \frac{I^{(0i)}}{u - \mathcal{D}_i} \qquad \mod U_{s,N}^{-}. \tag{4.15}$$

Analogously, in the symmetric case one has

$$T_N(u) \equiv 1 + \sum_{i=1}^{N} \frac{I^{(0i)}}{u + \mathcal{D}_i} \qquad \mod U_{s,N}^{+} \tag{4.16}$$

Note that the latter presentations can be equivalently used in the spaces $\Lambda_{\pm}^{s,N}$ of S_N – (skew)invariants, since the RHS of 4.16 commutes with the total action of the symmetric group. In components it is given by the formula

$$t_{ab}(u) = \delta_{ab} + \sum_i \frac{E_{ab,i}}{u \pm \mathcal{D}_i^{(N)}}, \tag{4.17}$$

where $E_{ab,i}$ is the action of the Lie algebra $\mathfrak{gl}_s$ on i-th tensor copy of $V^{\otimes N}$

$$E_{ab,i}\Big(\ldots \otimes \underbrace{(e^c \otimes x^k)}_{i} \otimes \ldots\Big) = \delta_{bc}\Big(\ldots \otimes \underbrace{(e^a \otimes x^k)}_{i} \otimes \ldots\Big).$$

4.1 Spin Calogero-Sutherland system

The phase space of the quantum spin Calogero-Sutherland (CS) system consists of functions with values in vector space $(\mathbb{C}^s)^{\otimes N}$ while the dependence on spin in the Hamiltonian given by (1.1) is implicit [23]. Using the same recipe as after formula (1.1) and choosing the parameter $\alpha = \beta^{-1}$ more common in mathematical literature, we arrive after simple rescaling to the effective Hamiltonian

$$H = \alpha \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i} \right)^2 + \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right) - 2 \sum_{i<j} \frac{x_i x_j}{(x_i - x_j)^2} (1 - K_{ij}) ,$$

It acts in the space $V^{\otimes N}$ and we restrict this action to the space of total (skew)invariants $\Lambda_{\pm}^{s,N}$ (4.5). The Hamiltonian is expressed in terms of elements ε_i of the degenerate affine Hecke algebra by the following formula:

$$H = \sum_i \left(\varepsilon_i^2 - \alpha \varepsilon_i \right) ,$$

Due to commutativity $[\varepsilon_i, \varepsilon_j] = 0$ the Hamiltonian H belongs to the center of degenerate affine Hecke algebra.

The spin Calogero-Sutherland model admits Yangian symmetry, namely the action (4.17) of the Yangian on the space $\Lambda_{\pm}^{s,N}$ commutes with the Hamiltonian. Then the higher Hamiltonians of spin CS system can be chosen as coefficients of the quantum determinant which generate the center of the $\mathrm{Y}(\mathfrak{gl}_s)$. In fact one can choose any commutative subalgebra of $\mathrm{Y}(\mathfrak{gl}_s)$ including H to be the higher Hamiltonians, or the elements of center of the degenerate affinne Hecke algebra, for example

$$H_n = \sum_i \varepsilon_i^n . \tag{4.18}$$

Our main goal is to construct the limit of the above Yangian action when N tends to infinity. In particular, we get the limits of the above commuting family of Hamiltonians.

4.2 Projective properties of Yangian action

To construct the limit we need investigate the projective properties of the Yangian actions in phase spaces $\Lambda_{\pm}^{s,N}$ of CS model. For such purposes we use the multiplicative presentation (4.13) in terms commutative family of Dunkl operators. Such an analysis was done by D.Uglov in [57], but our description differs from that of [57].

The rings Λ_+^N ($\equiv \Lambda_+^{1,N}$ in the notations 4.5) of scalar symmetric functions form the projective system with respect to the maps

$$\omega_N^+ : \Lambda_+^N \to \Lambda_+^{N-1}, \qquad \omega_N^+ f(x_1, \ldots, x_N) = f(x_1, \ldots, x_{N-1}, 0). \tag{4.19}$$

Analogously, the spaces Λ_-^N ($\equiv \Lambda_-^{1,N}$ in our notations) of scalar skewsymmetric functions form the projective system with respect to the maps

$$\omega_N^- : \Lambda_-^N \to \Lambda_-^{N-1}, \qquad \omega_N^- f(x_1, \ldots, x_N) = (x_1 \ldots x_{N-1})^{-1} f(x_1, \ldots, x_{N-1}, 0). \tag{4.20}$$

The latter can be generalized to the spin case. Regard an element f of $\Lambda_-^{s,N}$ as $(\mathbb{C}^s)^{\otimes N}$ valued function $f = f(x_1, x_2, \dots, x_N)$. Set

$$\omega_N^-(f) = (x_1 \cdots x_{N-s})^{-1} \left(1^{\otimes(N-s)} \otimes e_1^\perp \otimes e_2^\perp \cdots \otimes e_s^\perp\right) f(x_1, \dots, x_{N-s}, 0, \dots 0), \quad (4.21)$$

which coincides with (4.20) in case $s = 1$. In components,

$$\omega_N^-(x_1^{a_1} e_{c_1} \otimes \ldots \otimes x_N^{a_N} e_{c_N}) = \delta_{a_N,0} \cdots \delta_{a_{N-s+1},0} \delta_{c_N,s} \cdots \delta_{c_{N-s+1},1} x_1^{a_1-1} e_{c_1} \otimes \ldots \otimes x_{N-s}^{a_{N-s}-1} e_{c_{N-s}}.$$

One can see that ω_N is a linear map from $\Lambda_-^{s,N}$ to $\Lambda_-^{s,N-s}$.

For the analysis of compatibility of transfer matrices with projection maps 4.19 and 4.20 we use Dunkl operators $d_i^{(N)}$ (now we use the upper index to distinguish the number of variables on which this operator acts).

Consider first the scalar case $s = 1$. Set

$$\begin{aligned}
A_N^{(N)} &= \alpha \frac{\partial}{\partial x_N} + \sum_{j<N} \frac{1}{x_N - x_j}(1 - K_{Nj}), \\
A_i^{(N)} &= \frac{1}{x_i - x_N}(1 - K_{iN}), \\
B_i^{(N)} &= d_i^{(N-1)} + 1, \qquad i < N
\end{aligned} \quad (4.22)$$

Note that operators $A_i^{(N)}$ and $B_i^{(N)}$ transform polynomials to polynomials and

$$[x_N, B_i^{(N)}] = 0 \quad (4.23)$$

The following statement is straightforward result of the analysis of 4.8:

Lemma 4.1 *The Dunkl operators $d_i^{(N)}$ admit the decomposition*

$$d_N^{(N)} = x_N A_N^{(N)}, \quad (4.24)$$

$$d_i^{(N)} = x_N A_i^{(N)} + B_i^{(N)} = x_N A_i^{(N)} + d_i^{(N-1)} + 1, \qquad i < N. \quad (4.25)$$

Relations 4.14, 4.24, 4.25 and 4.22 imply the compatibility relations of transfer matrices for N and $N-1$ variables in scalar case.

Proposition 4.1 *(i) In scalar symmetric case we have the following identity of operators from $\tilde{\Lambda}_+^N[u^{-1}] \to \tilde{\Lambda}_+^{N-1}[u^{-1}]$:*

$$\omega_N^+ T_N(u) = \frac{u+1}{u} T_{N-1}(u+1) \omega_N^+; \quad (4.26)$$

(ii) In scalar skewsymmetric case the following identity of operators from $\tilde{\Lambda}_-^N[u^{-1}] \to \tilde{\Lambda}_-^{N-1}[u^{-1}]$ holds:

$$\omega_N^- T_N(u) = \frac{u+1}{u} T_{N-1}(u - \alpha - 1) \omega_N^-. \quad (4.27)$$

Proof. Due to 4.24, any power of the operator $d_N^{(N)}$ is divisible by x_N so that the application of $\omega_N^\pm$ to $\left(1 + \dfrac{I^{(0N)}}{u \mp d_N}\right)$ reduces to the multiplication by the scalar operator

$\dfrac{u+1}{u}$. Next, in symmetric case for any $i<N$ due to 4.25 and 4.23 the action of any power of $d_i^{(N)}$ modulo ideal generated by x_N differs from the action of $d_i^{(N-1)}$ by shift by 1. This gives 4.26. In skewsymmetric case the action of $x_i\dfrac{\partial}{\partial x_i}$ on the product $(x_1\cdots x_{N-1})^{-1}$, see 4.21, gives additional shift by α. So we have 4.27. □

Iterating the relations 4.26 we see, that in symmetric case the renormalized transfer matrices

$$\tilde{T}_N(u)=\frac{u-N}{u}T_N(u-N) \tag{4.28}$$

are compatible with projection maps ω_N^+,

$$\omega_N^+\tilde{T}_N(u)=\tilde{T}_{N-1}(u)\omega_N^+. \tag{4.29}$$

In antisymmetric case we can use

$$\bar{T}_N(u)=f_N(u)T_N(u+\gamma N), \qquad \omega_N^-\bar{T}_N(u)=\bar{T}_{N-1}(u)\omega_N^-, \tag{4.30}$$

where $\gamma=\alpha+1$ and

$$f_N(u)=\prod_{k=1}^{N}\frac{u+k\gamma}{u+k\gamma+1}. \tag{4.31}$$

The statement of Proposition 4.1 can be generalized to skewsymmetric spin case.

Proposition 4.2 *The following identities of operators from* $\mathbb{C}^s\otimes\tilde{\Lambda}_-^{s,Ns}[u^{-1}]\to\mathbb{C}^s\otimes\tilde{\Lambda}_-^{s,(N-1)s}[u^{-1}]$ *holds:*

$$\omega_{Ns}^-T_{Ns}(u)=\frac{u+1}{u}T_{(N-1)s}(u-\alpha-s)\omega_{Ns}^-. \tag{4.32}$$

Proof. The proof of Proposition 4.2 distinguishes from the proof of Proposition 4.1 in two details. Set

$$T_{Ns}(u)=T'(u)T''(u),$$

where

$$T'(u)=\left(1+\frac{I_{Ns}}{u-d_{Ns}^{(Ns)}}\right)\cdots\left(1+\frac{I_{(N-1)s+1}}{u-d_{(N-1)s+1}^{(Ns)}}\right),$$
$$T''(u)=\left(1+\frac{I_{(N-1)s}}{u-d_{(N-1)s}^{(Ns)}}\right)\cdots\left(1+\frac{I_1}{u-d_1^{(Ns)}}\right)$$

Then modulo the ideal generated by $x_{(N-1)s+1},\ldots,x_{Ns}$ the action of each element $d_k^{(Ns)}$ in $T''(u)$ differs from that of $d_k^{((N-1)s)}$ in $T_{(N-1)s}(u)\omega_{Ns}^-$ by $s+\alpha$. This explains the shift of the spectral parameter. On the other hand the each $d_k^{(Ns)}$ in $T'(u)$ can be presented in a form

$$d_k^{(Ns)}=\sum_{j=1}^{s}x_{(N-1)s+j}A_{jk}, \qquad k>(N-1)s$$

where A_{jk} trasform polynomials to polynomials. Thus the computation of the action of $\omega^-_{Ns} T'(u)$ reduces to the identity

$$\left(1 + \frac{I_{Ns}}{u}\right) \cdots \left(1 + \frac{I_{(N-1)s+1}}{u-s+1}\right) e_1 \otimes \ldots e_s \equiv \frac{u+1}{u} e_1 \otimes \ldots e_s \tag{4.33}$$

in the space $\left(\operatorname{End} \mathbb{C}^s \otimes \tilde{\Lambda}^{s,s}_-\right)$, equivalent to the computation of the q - determinant of Yangian matrix [36]: We demonstrate 4.33 for $s = 2$, renaming tensor indices and assuming final projection to $\tilde{\Lambda}^{s,Ns}_-$:

$$\left(1 + \frac{\sum_{ij=1}^2 E^{(0)}_{ij} \otimes E^{(1)}_{ij}}{u}\right) \left(1 + \frac{\sum_{kl=1}^2 E^{(0)}_{kl} \otimes E^{(2)}_{kl}}{u-1}\right) e_1 \otimes e_2 \equiv$$
$$\left(1 + \frac{E^{(0)}_{11} \otimes E^{(1)}_{11}}{u} + \frac{E^{(0)}_{22} \otimes E^{(2)}_{22}}{u-1} + \frac{E^{(0)}_{21} E^{(0)}_{12} \otimes E^{(1)}_{21} \otimes E^{(2)}_{12}}{u(u-1)}\right) e_1 \otimes e_2 \equiv$$
$$\left(1 + \frac{E^{(0)}_{11}}{u} + \frac{E^{(0)}_{22}}{u-1} - \frac{E^{(0)}_{22}}{u(u-1)}\right) \otimes e_1 \otimes e_2 \equiv$$
$$\left(1 + \frac{E^{(0)}_{11}}{u} + \frac{E^{(0)}_{22}}{u}\right) \otimes e_1 \otimes e_2 = \frac{u+1}{u} \otimes e_1 \otimes e_2.$$

□

Set $\gamma = \alpha + s$ and

$$\bar{T}_{Ns}(u) = f_N(u) T_{Ns}(u + \gamma N), \tag{4.34}$$

where now

$$f_N(u) = \prod_{k=1}^{N} \frac{u + k\gamma}{u + k\gamma + 1} = \frac{\Gamma\left(\frac{u}{\gamma} + N + 1\right) \Gamma\left(\frac{u+1}{\gamma} + 1\right)}{\Gamma\left(\frac{u+1}{\gamma} + N + 1\right) \Gamma\left(\frac{u}{\gamma} + 1\right)}$$

treated as asymptotical series in u^{-1}. Then $\bar{T}_{Ns}(u)$ satisfy compatibility conditions

$$\omega^-_{Ns} \bar{T}_{Ns}(u) = \bar{T}_{(N-1)s} \omega^-_{Ns}$$

and form a projective system of transfer matrices.

5 Bosonic limit of spin Calogero-Sutherland system

In this section we observe the results of [27] using slightly different language.

Denote by $\Lambda^{(s)}$ the free unital associative commutative algebras generated by the elements

$$p_{a,k}, \qquad a = 1, \ldots, s, \; k = 1, 2, \ldots$$

The ring $\Lambda^{(s)}$ can be viewed as the ring of polysymmetric functions, that is the projective limit of polynomial functions on the variables $x_{11}, \ldots x_{1n_1}, x_{21}, \ldots x_{2n_2}, \ldots x_{s1}, \ldots x_{sn_s}$, symmetric on each group of variables $x_{a1}, \ldots x_{an_a}$, $a = 1, ..., s$. Here $p_{a,k}$ corresponds to the Newton sums $x_{a1}^k + x_{a2}^k + \cdots$. Denote by $\hat{\Lambda}^{(s)}$ the free unital associative commutative algebras generated by the elements

$$p_{a,k}, \qquad a = 1, \ldots, s, \; k = 0, 1, 2, \ldots$$

We have $\hat{\Lambda}^{(s)} \supset \Lambda^{(s)}$. Additional "zero modes" $p_{a,0}$ will further serve to count the numbers of variables in each group.

Let $\mathcal{H}^s$ be the Heisenberg algebra with generators $a_{c,k}, c = 1, \ldots, s, \; k = 0, 1, ...$ and $(q_c)^{\pm 1}$, which satisfy the relations

$$[a_{c,k}, a_{d,l}] = k\delta_{cd}\delta_{k,-l}, \qquad q_c a_{d,k} = (a_{d,k} + \delta_{cd}\delta_{k0})q_c. \tag{5.1}$$

The space $\hat{\Lambda}^{(s)}$ is a representation of the Heisenberg algebra $\mathcal{H}^s$, where

$$\begin{array}{lll} a_{c,k} & \mapsto & p_{c,-k}, \quad k \leq 0, \\ a_{c,k} & \mapsto & k\frac{\partial}{\partial p_{c,k}}, \quad k > 0, \end{array}, \qquad q_c \mapsto e^{\frac{\partial}{\partial p_{c,0}}}.$$

The unit of the ring $\hat{\Lambda}^{(s)}$ is then identified with the vacuum vector $|0\rangle_+$, so that

$$a_{c,k}|0\rangle_+ = 0, \; c = 1, ..., s, \quad k > 0, \qquad q_c|0\rangle_+ = |0\rangle_+, \; c = 1, ..., s. \tag{5.2}$$

Denote by ${}_+\langle 0|$ the vector of the dual space, which satisfies the relations

$${}_+\langle 0|a_{c,k} = 0, \qquad c = 1, ..., s, \quad k \leq 0. \tag{5.3}$$

For $c = 1, ..., s$ denote by $\varphi_c^-(z)$ the series

$$\varphi_c^-(z) = \sum_{n \leq 0} a_{c,n} z^n \tag{5.4}$$

and by $e_c^\perp$ the linear operator $\mathbb{C}^s \to \mathbb{C}$ given by the relation

$$e_c^\perp(e_b) = \delta_{bc}.$$

Define linear operators

$$\begin{array}{lll} \Phi_c(z) = \exp\left(\sum_{n>0} \frac{a_{c,n}}{n} z^n\right) q_c : & \hat{\Lambda}^{(s)} \to \hat{\Lambda}^{(s)} \otimes \mathbb{C}[z], & c = 1, ..., s, \\ \mathbf{\Phi}(z) = \sum_c \Phi_c(z) \otimes e_c : & \hat{\Lambda}^{(s)} \to \hat{\Lambda}^{(s)} \otimes V, & \\ \mathbf{\Phi}^*(z) = \sum_c \varphi_c^-(z) \cdot \Phi_c^{-1}(z) \otimes e_c^\perp : & \hat{\Lambda}^{(s)} \otimes V \to \hat{\Lambda}^{(s)} \otimes \mathbb{C}[z]. & \end{array}$$

For instance, for any $|v\rangle_+ \in \hat{\Lambda}^{(s)}$

$$\mathbf{\Phi}^*(z)(|v\rangle_+ \otimes z^k \otimes e_c) = z^k \varphi_c^-(z)\Phi_c^{-1}(z)|v\rangle_+,$$

where

$$\Phi_c^{-1}(z) = q_c^{-1} \exp \sum_{n>0} -\frac{a_{c,n}}{n} z^n.$$

For any $|v\rangle_+ \in \hat{\Lambda}^{(s)}$ consider the matrix element $\tilde{\pi}_N(|v\rangle_+) \in V^{\otimes N}$,

$$\tilde{\pi}_N(|v\rangle_+) = {}_+\langle 0|(\mathbf{\Phi}(z_N) \otimes 1^{\otimes(N-1)}) \cdots (\mathbf{\Phi}(z_2) \otimes 1)\mathbf{\Phi}(z_1)|v\rangle_+$$

which we shortly denote by

$$\tilde{\pi}_N(|v\rangle_+) = {}_+\langle 0|\mathbf{\Phi}(z_N)\mathbf{\Phi}(z_2) \cdots \mathbf{\Phi}(z_1)|v\rangle_+ \tag{5.5}$$

In components,

$$\tilde{\pi}_N(|v\rangle_+) = \sum_{c_1,..,c_N=1}^{s} {}_+\langle 0|\Phi_{c_N}(z_N) \cdots \Phi_{c_1}(z_1)|v\rangle_+ \cdot e_{c_N} \otimes \ldots \otimes e_{c_1}.$$

The commutativity

$$\Phi_b(z_1)\Phi_c(z_2) = \Phi_c(z_2)\Phi_b(z_1) \tag{5.6}$$

implies that the matrix element 5.5 belongs to the space $\Lambda_+^{s,N}$. Indeed,

$$\sigma_{ij}\left(\sum_{c_1,..,c_N=1}^{s} {}_+\langle 0|\cdots \Phi_{c_j}(z_j)\cdots\Phi_{c_i}(z_i)\cdots|v\rangle_+ \cdot \ldots \otimes e_{c_j} \otimes \ldots \otimes e_{c_i} \otimes \ldots\right) =$$
$$= \sum_{c_1,..,c_N=1}^{s} {}_+\langle 0|\cdots \Phi_{c_j}(z_i)\cdots\Phi_{c_i}(z_j)\cdots|v\rangle_+ \cdot \ldots \otimes e_{c_i} \otimes \ldots \otimes e_{c_j} \otimes \ldots =$$
$$= \sum_{c_1,..,c_N=1}^{s} {}_+\langle 0|\cdots \Phi_{c_i}(z_i)\cdots\Phi_{c_j}(z_j)\cdots|v\rangle_+ \cdot \ldots \otimes e_{c_j} \otimes \ldots \otimes e_{c_i} \otimes \ldots$$

In the last equality we change the indices of summation c_i by c_j.

Our goal is to pull back the Yangian action 4.17 in $\Lambda_+^{s,N}$ through the map $\tilde{\pi}_N$. The dissection of the relation 4.17 shows that the application of each Yangian generator to a vector $|v\rangle_+ \in \Lambda_+^{s,N}$ can be decomposed into several steps. First we present the symmetric tensor $|v\rangle_+ \in \Lambda_+^{s,N}$ as an element of $(\mathbb{C}[x_i] \otimes \mathbb{C}^s) \otimes \Lambda_+^{s,N-1}$ for each tensor component, producing an equivariant family of vectors, which can be completely described by the element of $V \otimes \Lambda_+^{s,N-1} \sim (\mathbb{C}[x_i] \otimes \mathbb{C}^s) \otimes \Lambda_+^{s,N-1}$ — the decomposition of $|v\rangle_+$ over the first tensor component. Then we apply the power of Heckman operator $\mathcal{D}_i^{(N)}$ to the i-th vector of this equivariant family and get another equivariant family. The last step is the symmetrization — the sum of all members of the equivariant family.

Denote by $\iota_N : \Lambda_+^{s,N} \to V \otimes \Lambda_+^{s,N-1}$ the decomposition of the symmetric tensor v over the first tensor component,

$$\iota_N\left(\sum_k f_{1k}(z) \otimes \cdots \otimes f_{Nk}(x_N)\right) = \sum_k f_{1k}(z) \otimes (f_{2k}(x_2) \otimes \cdots \otimes f_{Nk}(x_N))\,. \tag{5.7}$$

Here $f_{1k}(z)$ and $f_{jk}(x_k)$, $j > 1$ are $\mathbb{C}^s$ valued polynomials.

Lemma 5.1 . *We have the following equality of linear maps $\hat{\Lambda}^{(s)} \to \Lambda_+^{s,N}$:*

$$(\tilde{\pi}_{N-1} \otimes 1)\, \mathbf{\Phi}(z) = \iota_N \tilde{\pi}_N. \tag{5.8}$$

Proof. Applying both sides of 5.8 to a vector $|v\rangle_+ \in \hat{\Lambda}^{(s)}$ we get the tautology: both sides are equal to

$$_+\langle 0|\mathbf{\Phi}(x_N)\mathbf{\Phi}(x_{N-1})\cdots\mathbf{\Phi}(x_2)\mathbf{\Phi}(z)|v\rangle_+.$$

□

For each tensor $u \in V \otimes \Lambda_+^{s,N-1}$, symmetric with respect to diagonal permutations of all tensor factor except the first, denote by $E_N(u)$ its total symmetrization

$$E_N(u) = \sum_{j=1}^{N} \sigma_{1j}(u), \tag{5.9}$$

where $\sigma_{ij} = K_{ij}P_{ij}$ is the permutation of i-th and j-th tensor factors. On the other hand, for each $\boldsymbol{F}(z) \in \hat{\Lambda}^{(s)} \otimes V$ define the element $\mathcal{S}(\boldsymbol{F}(z)) \in \hat{\Lambda}^{(s)}$ as the formal integral

$$\mathcal{S}(\boldsymbol{F}(z)) = \frac{1}{2\pi i}\oint \frac{dz}{z}\mathbf{\Phi}^*(z)\boldsymbol{F}(z), \tag{5.10}$$

which counts zero term of the Laurent series. The following lemma establishe the map $\mathcal{S}$ as the pullback of the finite symmetrization. This is the crucial point of the construction.

Lemma 5.2 *For each $\boldsymbol{F}(z) \in \hat{\Lambda}^{(s)} \otimes V$ and any natural N we have the equality of elements of $\Lambda_+^{s,N}$:*

$$E_N(\tilde{\pi}_{N-1} \otimes 1)(\boldsymbol{F}(z)) = \tilde{\pi}_N \mathcal{S}(\boldsymbol{F}(z)). \tag{5.11}$$

Proof. Let $\boldsymbol{F}(z)$ has the form

$$\boldsymbol{F}(z) = \sum_{c=1}^{s} F_c(z) \otimes e_c, \qquad F_c(z) \in \hat{\Lambda}^{(s)} \otimes \mathbb{C}[z].$$

Consider first the LHS of 5.11. This is the symmetrization 5.9 of the tensor

$$\sum_{c_1,..,c_N=1}^{s} {}_+\langle 0|\Phi_{c_N}(x_N)\cdots\Phi_{c_2}(x_2)F_{c_1}(x_1)\cdot e_{c_N} \otimes \ldots \otimes e_{c_1},$$

which can be written by means of proper changes of summation indices as the sum

$$\sum_{k=1}^{N}\sum_{c_1,..,c_N=1}^{s} {}_+\langle 0|\Phi_{c_N}(x_N)\cdots\Phi_{c_{k+1}}(x_{k+1})\Phi_{c_{k-1}}(x_{k-1})\cdots\Phi_{c_2}(x_2)\cdots F_{c_k}(x_k)\cdot e_{c_N} \otimes \ldots \otimes e_{c_1}.$$

Inserting in each summand the corresponding product

$$1 = \Phi_{c_k}(x_k)\Phi_{c_k}^{-1}(x_k)$$

and using the commutativity (5.6) we rewrite it as

$$\sum_{k=1}^{N}\sum_{c_1,..,c_N=1}^{s} {}_+\langle 0|\prod_{j=1}^{N}\Phi_{c_j}(x_j)\cdot\Phi_{c_k}^{-1}(x_k)F_{c_k}(x_k)\cdot e_{c_N} \otimes \ldots \otimes e_{c_1} =$$

$$\sum_{k=1}^{N}\sum_{c_1,..,c_N=1}^{s} \frac{1}{2\pi i}\int_{z\circlearrowleft x_k} dz\ {}_+\langle 0|\prod_{j=1}^{N}\Phi_{c_j}(x_j)\cdot\frac{\Phi_{c_k}^{-1}(z)F_{c_k}(z)}{z - x_k}\cdot e_{c_N} \otimes \ldots \otimes e_{c_1}. \tag{5.12}$$

The RHS of 5.11 is

$$\frac{1}{2\pi i}{}_{+}\langle 0|\prod_{j=1}^{N}\boldsymbol{\Phi}(x_j)\oint\frac{dz}{z}\boldsymbol{\Phi}^*(z)\boldsymbol{F}(z).$$

In components it looks as

$$\frac{1}{2\pi i}\sum_{a=1}^{s}\sum_{c_1,..,c_N=1}^{s}{}_{+}\langle 0|\prod_{j=1}^{N}\Phi_{c_j}(x_j)\oint\frac{dz}{z}\varphi_a^-(z)\Phi_a^{-1}(z)F_a(z)\cdot e_{c_N}\otimes\ldots\otimes e_{c_1}. \tag{5.13}$$

The normal ordering of the above matrix elements assumes due to 5.3 the move of all $\varphi_a^-(z)$ to the left vacuum using the relation

$$\Phi_c(x)\varphi_a^-(z)=\left(\varphi_a^-(z)+\frac{\delta_{ac}}{1-\frac{x}{z}}\right)\Phi_c(x) \tag{5.14}$$

which follows from 5.1. In particular, the formal integral in 5.13 can be regarded as a contour integral, where the contour C of integration encloses all points x_j. Since

$${}_{+}\langle 0|\varphi_a^-(z)=0,$$

we arrive to the expression

$$\sum_{k=1}^{N}\sum_{c_1,..,c_N=1}^{s}\frac{1}{2\pi i}\int_{z\circlearrowleft x_k}{}_{+}\langle 0|\prod_{j=1}^{N}\Phi_{c_j}(x_j)\oint\frac{dz}{z}\frac{\Phi_{c_k}^{-1}(z)F_{c_k}(z)}{1-\frac{x_k}{z}}\cdot e_{c_N}\otimes\ldots\otimes e_{c_1},$$

which is identical to 5.12. □

We now apply statements of Lemma 5.1 and 5.2 for the construction of a pullback of the Dunkl operator.

Let $\tilde{\mathcal{D}}:\hat{\Lambda}^{(s)}\otimes V\to\hat{\Lambda}^{(s)}\otimes V$ be the linear map, such that for any $\boldsymbol{F}(z)\in\hat{\Lambda}^{(s)}\otimes V$

$$\tilde{\mathcal{D}}\boldsymbol{F}^{(1)}(z)=\alpha z\frac{d}{dz}\boldsymbol{F}^{(1)}(z)+\frac{z}{2\pi i}\oint\frac{d\xi}{\xi^2(1-\frac{z}{\xi})}\boldsymbol{\Phi}^{*(2)}(\xi)\boldsymbol{\Phi}^{(2)}(z)\boldsymbol{F}^{(1)}(\xi) \tag{5.15}$$

Here the upper index (i), $i=1,2$ indicates in which tensor copy of $\mathbb{C}^s$ the corresponding vector lives or an operator acts. In components,

$$\tilde{\mathcal{D}}(F_a(z)\otimes e_a)=\left(\alpha z\frac{d}{dz}F_a(z)+\frac{z}{2\pi i}\oint\sum_{c=1}^{s}\frac{d\xi}{\xi^2(1-\frac{z}{\xi})}\varphi_c^-(\xi)\Phi_c^{-1}(\xi)\Phi_c(z)F_a(\xi)\right)\otimes e_a$$

We state that the operator $\mathcal{D}$ is the pullback of the equivariant family of Heckman operators $\mathcal{D}_i^{(N)}$.

Proposition 5.1 *For any $\boldsymbol{F}(z)\in\hat{\Lambda}^{(s)}\otimes V$ we have*

$$(\tilde{\pi}_{N-1}\otimes 1)\,\tilde{\mathcal{D}}(\boldsymbol{F}(x_1))=\mathcal{D}_1^{(N)}\,(\tilde{\pi}_{N-1}\otimes 1)\,\boldsymbol{F}(x_1) \tag{5.16}$$

Proof. The only nontrivial part is the pullback of the difference part of the Heckman operator. The difference part $\tilde{\mathcal{D}}_1^{(N)}$ of Heckman operator $\mathcal{D}_1^{(N)}$ in the space $V(x_1)\otimes\Lambda_+^{s,N-1}$, where $V(x_1)=\mathbb{C}[x_1]\otimes\mathbb{C}^s$, can be described as the composition of three operations. First we include into $V(x_1)\otimes V(x_2)\otimes \Lambda_+^{s,N-2}$ by means of $1\otimes\iota_{N-1}$, then apply the operator $\dfrac{1-K_{12}}{x_1-x_2}$ and finally sum up over all the variables except x_1 by means of the summation E_{N-1},

$$\tilde{\mathcal{D}}_1^{(N)} = E_{N-1}\ \circ\ \frac{1-K_{12}}{x_1-x_2}\ \circ\ 1\otimes\iota_{N-1}$$

The pullback of the inclusion $1\otimes\iota_{N-1}$ is $\mathbf{\Phi}^{(2)}(x_2)$ due to Lemma 5.1, the pullback of E_{N-1} is $\oint \mathbf{\Phi}^{*(2)}(x_2)\dfrac{dx_2}{x_2}$, the pullback of the operator $\dfrac{1-K_{12}}{x_1-x_2}$ is this very operator $\dfrac{1-K_{12}}{x_1-x_2}$. We see that the pullback of the difference operator $\tilde{\mathcal{D}}_1^{(N)}$ has the form

$$\tilde{\mathcal{D}}\boldsymbol{F}^{(1)}(x_1) = \frac{x_1}{2\pi i}\oint\frac{dx_2}{x_2}\mathbf{\Phi}^{*(2)}(x_2)\frac{\mathbf{\Phi}^{(2)}(x_2)\boldsymbol{F}^{(1)}(x_1)-\mathbf{\Phi}^{(2)}(x_1)\boldsymbol{F}^{(1)}(x_2)}{x_1-x_2} \tag{5.17}$$

Any matrix element of the ratio inside the integral is a polynomial on x_1 and x_2 and can be equally decomposed into a series either in the region $|x_1|<|x_2|$ or in the region $|x_1|>|x_2|$. In the region $|x_1|<|x_2|$ in the first integral

$$\frac{x_1}{2\pi i}\oint\frac{dx_2}{x_2}\mathbf{\Phi}^{*(2)}(x_2)\frac{\mathbf{\Phi}^{(2)}(x_2)\boldsymbol{F}^{(1)}(x_1)}{x_1-x_2} = \sum_{c=1}^{s}\frac{x_1}{2\pi i}\oint\frac{dx_2}{x_2}\varphi_c^-(x_2)\frac{\boldsymbol{F}^{(1)}(x_1)}{x_1-x_2}$$

we have only negative powers of x_2 and this integral vanish. Thus we get (5.15). □

Let $E_{ab}\in\operatorname{End}\mathbb{C}^s$, be the matrix unit, $E_{ab}(e_c)=\delta_{bc}e_a$. Denote by $\mathcal{E}_{ab}$, the operator $1\otimes 1\otimes E_{ab}:\hat{\Lambda}^{(s)}\otimes V\to\hat{\Lambda}^{(s)}\otimes V$:

$$\mathcal{E}_{ab}\boldsymbol{F}(z) = F_b(z)\otimes e_a.$$

For $a,b=1,...,s$ and $n=1,...$ set

$$T_{ab,n} = \frac{(-1)^n}{2\pi i}\oint\frac{dz}{z}\mathbf{\Phi}^*(z)\mathcal{E}_{ab}\tilde{\mathcal{D}}^n\mathbf{\Phi}(z) \tag{5.18}$$

Summarizing the statements above we get the following result [27]

Theorem 5.1 *The operator $T_{ab,n}$,see (5.18) is the pullback of the Yangian generator $t_{ab,n}$, see (4.17), (4.2):*

$$\tilde{\pi}_N T_{ab,n} = t_{ab,n}\tilde{\pi}_N \qquad \textit{for any} \qquad N\in\mathbb{N}.$$

In particular, the operators 5.18 form level zero representation of the Yangian $Y(\mathfrak{gl}_s)$ in $\hat{\Lambda}^{(s)}$. Here we use the property

$$\cap_{N\in\mathbb{N}}\operatorname{Ker}\tilde{\pi}_N = 0 \tag{5.19}$$

of the ring of symmetric functions which we assume to be known.

5.1 Hamiltonians

In this section we provide explicit expressions for the first few Hamiltonians constructed by means of the procedure (5.18). We present the expressions for the first $H_n=\sum_i\varepsilon_i^n$ in terms of the Yangian generators $t_{ab,k}$ in representation (4.17) :

$$H_1 = -\sum_a t_{aa,1} + \frac{1}{2}\sum_{a,b}t_{ab,0}t_{ba,0} - \frac{s}{2}\sum_a t_{aa,0}, \tag{5.20}$$

$$
\begin{aligned}
H_2 = \sum_a t_{aa,2} - \sum_{a,b} t_{ab,0} t_{ba,1} + s \sum_a t_{aa,1} + \frac{1}{3} \sum_{a,b,c} t_{ab,0} t_{bc,0} t_{ca,0} \\
-\frac{2s}{3} \sum_{a,b} t_{ab,0} t_{ba,0} + \frac{1}{6} \sum_{a,b} t_{aa,0} t_{bb,0} + \frac{2s^2-1}{6} \sum_a t_{aa,0},
\end{aligned}
\tag{5.21}
$$

where s is the number of spin variables, we mean that all summations are from 1 to s. On the other hand, the Hamiltonians can be obtained as elements of the q-determinant (4.4). Consider the representation of the q-determinant :

$$
\mathrm{q}\ \det t(u) = 1 + \frac{\Delta_0}{u} + \frac{\Delta_1}{u^2} + \frac{\Delta_2}{u^3} + \dots.
$$

The elements Δ_i can be expressed in terms of $t_{ab,k}$. Explicit expressions for $\Delta_0, \Delta_1, \Delta_2$ are presented in Appendix 3. The Hamiltonians can be rewritten in the following form:

$$
H_1 = -\Delta_1 + \frac{1}{2}\Delta_0^2 - \frac{1}{2}\Delta_0, \tag{5.22}
$$

$$
H_2 = \Delta_2 - \Delta_0\Delta_1 + \Delta_1 + \frac{1}{3}\Delta_0^3 - \frac{1}{2}\Delta_0^2 + \frac{1}{6}\Delta_0. \tag{5.23}
$$

Replacing $t_{ab,k}$ in (5.20),(5.21) by its representation (5.18) we obtain the expressions for the pullbacks $\mathscr{H}_1, \mathscr{H}_2$ of the first Hamiltonians (4.18) in $\hat{\Lambda}^{(s)}$:

Proposition 5.2 *The Hamiltonians $\mathscr{H}_1$ and $\mathscr{H}_2$ are the pullbacks of the Hamiltonians (4.18) and have the following form:*

$$
\mathscr{H}_1 = \alpha \sum_b \oint \frac{d\xi}{\xi} \varphi_b^-(\xi)\varphi_b^+(\xi) + \frac{1}{2}\left(a_0^2 - a_0\right), \tag{5.24}
$$

$$
\begin{aligned}
\mathscr{H} = \mathscr{H}_2 - \alpha\mathscr{H}_1 = \alpha^2 \sum_a \oint \frac{d\xi}{\xi} \varphi_a^-(\xi)(\varphi_a^+(\xi))^2 + \alpha(\alpha-1) \sum_a \oint \frac{d\xi}{\xi} \varphi_a^-(\xi)(\varphi_a^+(\xi))' + \\
-\alpha \sum_a \oint \frac{d\xi}{\xi} \varphi_a^-(\xi)\varphi_a^+(\xi) + \alpha \sum_{a,b} \oint \frac{d\xi}{\xi} \varphi_a^-(\xi)\varphi_b^-(\xi)\varphi_a^+(\xi) + \\
\alpha \sum_{a>b} \oint \frac{d\xi d\eta}{\xi\eta} \varphi_a^-(\eta)\varphi_b^-(\xi) \sum_{k\in\mathbb{N}} k \left(\frac{\xi^k}{\eta^k} + \frac{\eta^k}{\xi^k}\right) \Phi_a^{-1}(\eta)\Phi_b(\eta)\Phi_b^{-1}(\xi)\Phi_a(\xi).
\end{aligned}
\tag{5.25}
$$

Here $a_0 = \sum_{c=1}^s a_{c,0}$. The same expressions are presented in [7].

5.2 Classical limit

In this section we investigate the classical limit of the Hamiltonian (5.25). In the spinless case it leads to the periodic Benjamin-Ono equation as we explain in section 4.1. By introducing $\beta = \frac{1}{\alpha}$ and multiplying (5.25) by β^3, we obtain:

$$
\begin{aligned}
\beta^3\mathscr{H} = \beta \oint_\xi \frac{d\xi}{\xi} \varphi_a^-(\xi)(\varphi_a^+(\xi))^2 + (\beta - \beta^2) \oint_\xi \frac{d\xi}{\xi} \varphi_a^-(\xi)(\varphi_a^+(\xi))' + \beta^2 \sum_{a,b} \oint_\eta \frac{d\eta}{\eta} \varphi_a^-(\eta)\varphi_b^-(\eta)\varphi_a^+(\eta) \\
-\beta^2 \sum_a \oint \frac{d\xi}{\xi} \varphi_a^-(\xi)\varphi_a^+(\xi) + \beta^2 \sum_{a>b} \oint_{\eta,\xi} \frac{d\xi d\eta}{\xi\eta} \varphi_a^-(\eta)\varphi_b^-(\xi) \sum_k k \left(\frac{\xi^k}{\eta^k} + \frac{\eta^k}{\xi^k}\right) \Phi_{ba}(\eta)\Phi_{ab}(\xi),
\end{aligned}
\tag{5.26}
$$

where for convenience we denote $\Phi_a(x)\Phi_b^{-1}(x)$ by $\Phi_{ab}(x)$. Introducing the classical variables $\alpha_{n,a},\ n\in\mathbb{Z}$ with Poisson bracket relations

$$\{\alpha_{c,n},\alpha_{b,m}\} = n\delta_{c,b}\delta_{n+m,0}, \tag{5.27}$$

combine them into generating functions

$$\phi_a^-(\xi)=\sum_{n=0}^{\infty}\frac{\alpha_{a,-n}}{\xi^n},\ \ \phi_a^+(\xi)=\sum_{n=1}^{\infty}\alpha_{a,n}\xi^n, \qquad \text{and}\ \ \phi_a(x)=\phi_a^+(x)+\phi_a^-(x).$$

Then

$$\{\phi_a^-(x),\phi_b^+(y)\}=\delta_{a,b}\frac{y}{(1-\frac{y}{x})^2}, \qquad \{\phi_a^\pm(x),\phi_b^\pm(y)\}=0.$$

or

$$\{\phi_a(x),\phi_b(y)\}=\delta'\left(x/y\right)\delta_{a,b}.$$

Denote by $\mathcal{V}_a(\xi)$ the classical counterparts of the vertex operatosr $\Phi_a(\xi)$ satisfing the relations

$$\xi\frac{d\log\mathcal{V}_a(\xi)}{d\xi}=\phi_a^+(\xi),$$

and

$$\{\phi_a^-(x),\mathcal{V}_b(y)\}=-\delta_{a,b}\frac{\mathcal{V}_b(y)}{1-\frac{y}{x}}, \qquad \{\phi_a^+(x),\mathcal{V}_b(y)\}=0. \tag{5.28}$$

As before, we use the notation $\mathcal{V}_{ab}(\xi)$ for $\mathcal{V}_a(\xi)\mathcal{V}_b^{-1}(\xi)$. Set

$$\begin{aligned}\mathscr{H}^{\mathrm{cl}} = &\oint_\xi\frac{d\xi}{\xi}\phi_a^-(\xi)(\phi_a^+(\xi))^2+\oint_\xi\frac{d\xi}{\xi}\phi_a^-(\xi)(\phi_a^+(\xi))'+\sum_{a,b}\oint_\eta\frac{d\eta}{\eta}\phi_a^-(\eta)\phi_b^-(\eta)\phi_a^+(\eta)\\ &+\sum_{a>b}\oint_{\eta,\xi}\frac{d\xi d\eta}{\xi\eta}\phi_a^-(\eta)\phi_b^-(\xi)\sum_k k\left(\frac{\xi^k}{\eta^k}+\frac{\eta^k}{\xi^k}\right)\mathcal{V}_{ab}^{-1}(\eta)\mathcal{V}_{ab}(\xi).\end{aligned} \tag{5.29}$$

The operator $\mathscr{H}^{\mathrm{cl}}$ is the classical limit of the Hamiltonian (5.26) ($\beta\to 0$). The rule between the quantum commutator and Poisson bracket is $\beta^{-1}[\ ,\ ]\to\{\ ,\ \}$ and the classical variable $\phi_a(x)$ correpsonds to $\beta\varphi_a^-(x)+\varphi_a^+(x)$.

Proposition 5.3 [4] *The equations of motion determined by the Hamiltonian $\mathscr{H}^{cl}$ are:*

$$\begin{aligned}\{\phi_a^+(x),\mathscr{H}^{cl}\} &= x\frac{\partial}{\partial x}(\phi_a^+(x))^2+\left(x\frac{\partial}{\partial x}\right)^2(\phi_a^+(x))+\sum_b x\frac{\partial}{\partial x}\left(\phi_b^-(x)\phi_a^+(x)\right)^+ +\\ &+\sum_b x\frac{\partial}{\partial x}\left(\phi_b^-(x)\phi_b^+(x)\right)^+ +\sum_{b\neq a}x\frac{\partial}{\partial x}\left(\mathcal{V}_{ab}^{-1}(x)x\frac{\partial}{\partial x}\left(\left(\phi_b^-(x)\mathcal{V}_{ab}(x)\right)^+-\left(\phi_b^-(x)\mathcal{V}_{ab}(x)\right)^-\right)\right)^+ =\\ &x\frac{\partial}{\partial x}(\phi_a^+(x))^2+\left(x\frac{\partial}{\partial x}\right)^2(\phi_a^+(x))+2\sum_b x\frac{\partial}{\partial x}\left(\phi_b^-(x)\phi_b^+(x)\right)^+\\ &+2\sum_{b\neq a}x\frac{\partial}{\partial x}\left(\mathcal{V}_{ab}^{-1}(x)x\frac{\partial}{\partial x}\left(\phi_b^-(x)\mathcal{V}_{ab}(x)\right)^+\right),\end{aligned} \tag{5.30}$$

[4] For a formal series $f(z)=\sum_{n\in\mathbb{Z}}f_nz^n$ we denote by $f^+(z)$ the series $f^+(z)=\sum_{n\geq 1}f_nz^n=\oint\frac{zf(\xi)d\xi}{(1-z/\xi)\xi}$ and by $f^-(z)$ the series $f^-(z)=\sum_{n\leq 0}f_nz^n=\oint\frac{f(\xi)d\xi}{(1-\xi/z)\xi}$.

$$
\begin{aligned}
\{\phi_a^-(x), \mathscr{H}^{cl}\} = {}& 2x\frac{\partial}{\partial x}\left(\phi_a^-(x)\phi_a^+(x)\right)^- - \left(x\frac{\partial}{\partial x}\right)^2\left(\phi_a^-(x)\right) + \sum_b x\frac{\partial}{\partial x}\left(\phi_b^-(x)\phi_a^-(x)\right) \\
& + \sum_{b\neq a}\left(\phi_a^-(x)\mathcal{V}_{ab}^{-1}(x)x\frac{\partial}{\partial x}\left(\left(\phi_b^-(x)\mathcal{V}_{ab}(x)\right)^+ - \left(\phi_b^-(x)\mathcal{V}_{ab}(x)\right)^-\right)\right)^- \\
& - \sum_{b\neq a}\left(\phi_b^-(x)\mathcal{V}_{ab}(x)x\frac{\partial}{\partial x}\left(\left(\phi_a^-(x)\mathcal{V}_{ab}^{-1}(x)\right)^+ - \left(\phi_a^-(x)\mathcal{V}_{ab}^{-1}(x)\right)^-\right)\right)^-
\end{aligned} \tag{5.31}
$$

Remark Unlike the Yangian generators the Hamiltonian (5.25) does not contain dual zero modes q_c. The same holds for the classical limit, where we can freely use the operators $\tilde{\mathcal{V}}_c(\xi) = \exp\sum_{n\geq 1}\frac{\alpha_{c,n}}{n}\xi^n$ instead of $\mathcal{V}_c(\xi)$. The Hamiltonian and the equations of motion do not change, while the brackets (5.28) turn into

$$
\{\phi_a^-(x), \tilde{\mathcal{V}}_b(y)\} = -\delta_{a,b}\frac{y/x\tilde{\mathcal{V}}_b(y)}{1-\frac{y}{x}}, \qquad \{\phi_a^+(x), \tilde{\mathcal{V}}_b(y)\} = 0.
$$

The quantum system is integrable: it has an infinite number of integrals of motion that can be obtained from the q-determinant of the Yangian generator function $T_{ab}(u)$. It is natural to assume that the classical system is integrable as well. In particular, it should admit a Lax pair presentation. Consider the operators L and M:

$$
\begin{aligned}
Lf &= z\frac{\partial}{\partial z}f(z) + \sum_a \mathcal{V}_a(z)\left(\phi_a^-(z)\mathcal{V}_a^{-1}(z)f(z)\right)^+, \\
Mf &= \left(z\frac{\partial}{\partial z}\right)^2 f(z) + 2\sum_b\left(\phi_b^+(z)\phi_b^-(z)\right)^+ f(z) + 2\sum_b \mathcal{V}_b(z)z\frac{\partial}{\partial z}\left(\phi_b^-(z)\mathcal{V}_b^{-1}(z)f(z)\right)^+.
\end{aligned} \tag{5.32}
$$

They act on the space of analytic functions

$$
f(z) = f_0 + f_1 z + f_2 z^2 + \dots,
$$

where coefficients f_i are polynomials in $\alpha_{c,n}$, where $n < 0$.

Proposition 5.4 *The operators L and M (5.32) represent a Lax pair of the classical system (5.29):*

$$
\frac{dL}{dt} = [M, L].
$$

6 Fermionic limit for spin system

Let $\mathcal{H}^s_-$ be the algebra of s free fermion fields. It is generated by the elements ψ_{nc} and ψ^*_{nc}, where $n \in \mathbb{Z}$ and $c = 1, \ldots, s$, which subject the relations

$$\begin{aligned} &\psi_{an}\psi_{bm} + \psi_{bm}\psi_{an} = 0, \qquad \psi^*_{an}\psi^*_{bm} + \psi^*_{bm}\psi^*_{an} = 0, \\ &\psi_{an}\psi^*_{bm} + \psi^*_{bm}\psi_{an} = \delta_{ab}\delta_{n,-m}. \end{aligned} \tag{6.1}$$

The algebra $\mathcal{H}^s_-$ is graded with

$$\deg \psi_{cn} = \deg \psi^*_{cn} = -n. \tag{6.2}$$

The algebra $\mathcal{H}^s_-$ admits a family of commuting automorphisms $\hat{Q}_c$, $c = 1, ..., s$ given by the relations

$$\hat{Q}_c(\psi_{bn}) = \psi_{b,n-\delta_{bc}}, \qquad \hat{Q}_c(\psi^*_{bn}) = \psi^*_{b,n+\delta_{bc}}. \tag{6.3}$$

Let $\mathcal{F}^s$ be the left representations of $\mathcal{H}^s_-$, generated by the vacuum state $|0\rangle$, and $\mathcal{F}^{s*}$ be the right $\mathcal{H}^s_-$ -module generated by the vacuume state $\langle 0|$, such that

$$\langle 0|0\rangle = 1$$

and

$$\begin{aligned} &\psi_{cn}|0\rangle = \psi^*_{cm}|0\rangle = 0 \qquad c = 1, ..., s, \qquad n \geq 0,\ m > 0, \\ &\langle 0|\psi_{cn} = \langle 0|\psi^*_{cm}| = 0, \qquad c = 1, ..., s, \qquad n < 0,\ m \leq 0. \end{aligned} \tag{6.4}$$

We use the following fermionic normal ordering rule:

$$\vdots\psi^*_{cn}\psi_{dm}\vdots = \begin{cases} \psi^*_{cn}\psi_{dm}, & m \geq 0 \\ -\psi_{dm}\psi^*_{cn}, & m < 0 \end{cases} . \tag{6.5}$$

It is compatible with relations (6.4).

The automorphisms 6.3 define invertible linear maps Q_c and Q_c^{-1} of the Fock space to itself which are compatible with these automorphisms and anticommute for different indices c_1 and c_2:

$$\begin{aligned} &Q_c^{-1}(x|0\rangle) = \hat{Q}_c^{-1}(x)\psi^*_{c,0}|0\rangle, \qquad Q_c(x|0\rangle) = \hat{Q}_c(x)\psi_{c,-1}|0\rangle, \\ &\langle 0|Q_c^{-1} = \langle 0|\psi^*_{c,1}, \qquad\qquad\qquad \langle 0|Q_c = \langle 0|\psi_{c,0}, \end{aligned} \tag{6.6}$$

so that for any $x \in \mathcal{H}^s_-$ and $|v\rangle \in \mathcal{F}^s$ we have

$$\hat{Q}_c(x)|v\rangle = Q_c x Q_c^{-1}|v\rangle. \tag{6.7}$$

Indeed, for $|v\rangle = y|0\rangle$ the RHS of (6.7) equals

$$\begin{aligned} Q_c x Q_c^{-1}|v\rangle = Q_c x Q_c^{-1}\left(y|0\rangle\right) = Q_c\left(x\hat{Q}_c^{-1}(y)\psi^*_{c,0}|0\rangle\right) = \hat{Q}_c(x)y\psi^*_{c,1}\psi_{c,-1}|0\rangle = \\ = \hat{Q}_c(x)y\left(1 - \psi_{c,-1}\psi^*_{c,1}\right)|0\rangle = \hat{Q}_c(x)y|0\rangle = \hat{Q}_c(x)|v\rangle. \end{aligned}$$

In the following we use the distinguished product of s such maps and automorphisms

$$\hat{Q} := \hat{Q}_s \cdots \hat{Q}_1, \qquad Q = Q_s \cdots Q_1. \tag{6.8}$$

In particular,

$$\begin{aligned} \langle 0|Q^{-1} &= \langle 0|\psi^*_{s,1}\cdots\psi^*_{1,1}, & Q^{-1}|0\rangle &= \psi^*_{s,0}\cdots\psi^*_{1,0}|0\rangle, \\ \langle 0|Q &= \langle 0|\psi_{s,0}\cdots\psi_{1,0}, & Q|0\rangle &= \psi_{s,-1}\cdots\psi_{1,-1}|0\rangle. \end{aligned} \tag{6.9}$$

Denote by $\mathcal{H}^s_-(z)$ the space of Laurent series

$$\sum_{n\in\mathbb{Z}} h_n z^n \in \mathcal{H}^s_-(z), \tag{6.10}$$

where each coefficient h_n is a series

$$h_n = \sum_k a^{11}_k + \sum_{kl} a^{21}_k a^{22}_l + \sum_{klm} a^{31}_k a^{32}_l a^{33}_m + \dots$$

where a^{ij}_k are either ψ_{cn} or ψ^*_{cn} for some c and n, such that the matrix coefficient $\langle\xi|h_n|v\rangle$ is well defined for any $\xi\in\mathcal{F}^{s*}$ and $v\in\mathcal{F}^s$. [5] We can assume for instance all series h_n to be fermionic normal ordered according to 6.5, $h_n = \vdots h_n \vdots$. We also use the notation $\mathcal{F}^s(z)$ for the space $\mathcal{F}^s\otimes\mathbb{C}[z,z^{-1}]]$.

Let $\Psi_c(z)$ and $\Psi^*_c(z)$ be the following elements of $\mathcal{H}^s_-(z)$,

$$\Psi_c(z) = \sum_{n\in\mathbb{Z}} \psi_{cn} z^n, \qquad \Psi^*_c(z) = \sum_{n\in\mathbb{Z}} \psi^*_{cn} z^{n-1}. \tag{6.11}$$

The field $\Psi_c(z)$ is of total degree zero, and the field $\Psi^*_c(z)$ is of total degree -1, once we set $\deg z = 1$. The relations 6.4 imply the commutativity

$$\Psi_c(x)\Psi_d(y) + \Psi_d(y)\Psi_c(x) = \Psi^*_c(x)\Psi^*_d(y) + \Psi^*_d(y)\Psi^*_c(x) = 0 \tag{6.12}$$

and normal ordering rules

$$\begin{aligned} &\Psi_c(x)\Psi_d(y) = \vdots\Psi_c(x)\Psi_d(y)\vdots, \qquad \Psi^*_c(x)\Psi^*_d(y) = \vdots\Psi^*_c(x)\Psi^*_d(y)\vdots, \\ &\Psi_c(x)\Psi^*_d(y) = \vdots\Psi_c(x)\Psi^*_d(y)\vdots + \frac{\delta_{cd}}{y-x}, \qquad x<y, \\ &\Psi^*_c(x)\Psi_d(y) = \vdots\Psi^*_c(x)\Psi_d(y)\vdots + \frac{\delta_{cd}}{y-x} \qquad x<y. \end{aligned} \tag{6.13}$$

which imply the relation

$$\frac{1}{2\pi i}\int_{z\circlearrowleft w} \Psi_c(w)\Psi^*_c(z)dz = \frac{1}{2\pi i}\int_{z\circlearrowleft w} \Psi^*_c(w)\Psi_c(z)dz = 1. \tag{6.14}$$

One can also see that

$$\hat{Q}\left(\Psi_c(z)\right) = z\Psi_c(z), \qquad \hat{Q}\left(\Psi^*_c(z)\right) = z^{-1}\Psi^*_c(z), \qquad c=1,...,s.$$

Boson–fermion correspondence says that the space $\mathcal{F}^s$ is a representation of the affine Lie algebra $\widehat{\mathfrak{gl}}_s$ of level one. The degree $-n$ generators $E_{ab,n}$ of $\widehat{\mathfrak{gl}}_s$, where $a,b=1,...,s$ and $n\in\mathbb{Z}$ satisfy the relations

$$[E_{ab,n}, E_{cd,m}] = \delta_{bc}E_{ad,n+m} - \delta_{ad}E_{cb,n+m} + n\delta_{n,-m}\delta_{ad}\delta_{bc}$$

[5] For instance the monomial $z^n\left(\sum_{k>0}\psi^*_{ck}\psi_{c,-k}\right)\notin\mathcal{H}^s_-(z)$.

They are presented in End $\mathcal{F}^s$ by operators

$$E_{ab,n} = \sum_{k+l=n} \vdots\psi^*_{al}\psi_{bk}\vdots, \tag{6.15}$$

where $\vdots\;\vdots$ means fermionic normal ordering (6.5). The generators

$$a_{b,n} := E_{bb,n}$$

form the Heisenberg algebra $\mathcal{H}^[s]$,

$$[a_{b,n}, a_{c,m}] = n\delta_{b,c}\delta_{n,-m}$$

so that

$$a_{c,k}|0\rangle = 0, \qquad \langle 0|a_{c,l} = 0, \qquad c = 1, ..., s, \quad k \geq 0, l \leq 0.$$

On the other side of boson–fermion correspondence we have the relations:

$$\Psi_c(z) = z^{a_{c,0}} \exp\left(\sum_{n<0} \frac{a_{c,n}}{n} z^n\right) \exp\left(\sum_{n>0} \frac{a_{c,n}}{n} z^n\right) Q_c$$

$$\Psi^*_c(z) = z^{-a_{c,0}} \exp\left(-\sum_{n<0} \frac{a_{c,n}}{n} z^n\right) \exp\left(-\sum_{n>0} \frac{a_{c,n}}{n} z^n\right) Q_c^{-1}.$$

The element

$$a_0 = \sum_{c=1}^{s} a_{c,0} = \sum_{c=1}^{s}\sum_{k\in\mathbb{Z}} \vdots\psi^*_{ck}\psi_{c,-k}\vdots \tag{6.16}$$

is central in $\widehat{\mathfrak{gl}}_s$ and satisfies the relation

$$Qa_0Q^{-1} = a_0 + s.$$

The Fock space $\mathcal{F}^s$ admits the orthogonal decomposition into direct sum of eigenspaces of operator a_0,

$$\mathcal{F}^s = \oplus_{N\in\mathbb{Z}}\mathcal{F}^s_N, \qquad \text{where} \qquad \mathcal{F}^s_N = \{|v\rangle \in \mathcal{F}^s : \; a_0|v\rangle = N|v\rangle\}. \tag{6.17}$$

The relation 6.17 implies that

$$Q\mathcal{F}^s_N = \mathcal{F}^s_{N-s}. \tag{6.18}$$

In the following we use the notation τ_N for the projection of $\mathcal{F}^s$ to $\mathcal{F}^s_N$ parallel to other eigenspaces of a_0:

$$\tau_N|v\rangle) = \delta_{N,k} \cdot |v\rangle \quad \text{for} \quad |v\rangle \in \mathcal{F}^s_k. \tag{6.19}$$

Let $\mathbf{\Psi}(z)$ and $\mathbf{\Psi}^*(z)$ be the following elements of $\mathcal{H}^s_-(z) \otimes \mathbb{C}^s$ and $\mathcal{H}^s_-(z) \otimes \mathbb{C}^{s*}$ correspondingly,

$$\mathbf{\Psi}(z) = \sum_c \Psi_c(z) \otimes e_c \qquad\qquad \mathbf{\Psi}^*(z) = \sum_c \Psi^*_c(z) \otimes e_c^{\perp}. \tag{6.20}$$

The field $\mathbf{\Psi}(z)$ defines the map from $\mathcal{F}^s$ to $\mathcal{F}^s(z) \otimes \mathbb{C}^s$,

$$\mathbf{\Psi}(z)|v\rangle = \sum_c \Psi_c(z)|v\rangle \otimes e_c,$$

which we denote by the same symbol $\mathbf{\Psi}(z)$. The field $\mathbf{\Psi}^*(w)$ defines a map from $\mathcal{H}^s_-(z)\otimes\mathbb{C}^s$ to $\mathcal{H}^s_-(z,w)$,

$$\mathbf{\Psi}^*(w)\left(\sum_c F_c(z)\otimes e_c\right)=\sum_c \Psi^*_c(w)F_c(z),$$

where $\mathcal{H}^s_-(z,w)$ is defined in the same way as $\mathcal{H}^s_-(z)$ (6.10). Here we regard $e_c^\perp$ as the linear map $e_c^\perp:\mathbb{C}^s\to\mathbb{C}$, such that $e_c^\perp(e_d)=\delta_{cd}$.

For any $|v\rangle\in\mathcal{F}^s$ consider the matrix element

$$\pi_N(|v\rangle)=\langle 0|(\mathbf{\Psi}(z_N)\otimes 1^{\otimes(N-1)})\cdots(\mathbf{\Psi}(z_2)\otimes 1)\mathbf{\Psi}(z_1)|v\rangle,$$

which we shortly denote by

$$\pi_N(|v\rangle)=\langle 0|\mathbf{\Psi}(z_N)\mathbf{\Psi}(z_2)\cdots\mathbf{\Psi}(z_1)|v\rangle. \tag{6.21}$$

In components,

$$\pi_N(|v\rangle)=\sum_{c_1,..,c_N=1}^{s}\langle 0|\Psi_{c_N}(z_N)\cdots\Psi_{c_1}(z_1)|v\rangle\cdot e_{c_1}\otimes\ldots\otimes e_{c_N}.$$

The commutativity 6.12 and the properties of the left vacuum 6.4 imply that the matrix element 6.21 belongs to the space $\Lambda_-^{s,N}$. Note that the map π_N factors through the projection τ_N (6.19),

$$\pi_N=\pi_N\tau_N$$

and equals zero for any $\mathcal{F}^s_M$ with $M\neq N$.

We are going now to construct the pullback through the maps π_N of the components of the Yangian generators.

Denote by $\iota_N:\Lambda_-^{s,N}\to\mathbb{C}^s[z]\otimes\Lambda_-^{s,N-1}$ the decomposition of the antisymmetric tensor v over the first tensor component, given by the relation 5.7. Denote by $\pi_{N-1,1}:\big(\mathcal{H}^s_-(z)\otimes\mathbb{C}^s\big)\otimes\mathcal{F}^s\to\mathbb{C}[x_2,...,x_N,z,z^{-1}]\otimes\mathbb{C}^{s\otimes N}$ the map defined as

$$\pi_{N-1,1}(\boldsymbol{F}(z)\otimes|v\rangle)=\langle 0|\mathbf{\Psi}(x_N)\cdots\mathbf{\Psi}(x_2)\boldsymbol{F}(z)|v\rangle.$$

Lemma 6.1 . *We have the following equality of linear maps $\mathcal{F}^s\to\Lambda_-^{s,N}$:*

$$\pi_{N-1,1}\mathbf{\Psi}(z)=\iota_N\pi_N. \tag{6.22}$$

Proof. This is again a tautology like in the proof of Lemma 5.1. □

For each polynomial tensor $\mathbb{C}^s[z]\in V\otimes\Lambda_-^{s,N-1}$, antisymmetric with respect to diagonal permutations of all tensor factor except the first, denote by $\mathrm{A}_N(u)$ its total (nonnormalized) antisymmetrization

$$\mathrm{A}_N(u)=u-\sum_{j=2}^{N}\sigma_{1j}(u). \tag{6.23}$$

On the other hand, for each $\boldsymbol{F}(z)\in\mathcal{H}^s_-(z)\otimes\mathbb{C}^s$ define the element $\mathcal{A}(\boldsymbol{F}(z))\in\mathcal{H}^s_-$ as the integral

$$\mathcal{A}(\boldsymbol{F}(z))=\frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0}dz\int_{u\circlearrowleft z}du\frac{\mathbf{\Psi}^*(u)\boldsymbol{F}(z)}{u-z}. \tag{6.24}$$

Remark. The integral over z is actually formal. The form 6.24 indicates the following. Assume that $\boldsymbol{F}(z)$ depends on a parameter w. Then the contour C of integration over z should not enclose the point $z = w$. One can always assume the condition $|w| > |z|$.

Let an element $\boldsymbol{F}(z) \in \mathcal{H}^s_-(z) \otimes \mathbb{C}^s$ satisfies the following conditions:

$$(i) \qquad \pi_{N,1}(\boldsymbol{F}(z) \otimes |v\rangle) \qquad \text{is a polynomial on } z \text{ for any } N \in \mathbb{N},\ v \in \mathcal{F}^s \tag{6.25}$$

$$(ii) \qquad \deg \boldsymbol{F}(z) = 0 \tag{6.26}$$

Here we assume that $\deg e_c = 0$ for any $e_c \in \mathbb{C}^s$.

The following lemma establishes the map $\mathcal{A}$ as the pullback of the finite antisymmetrization. This is the crucial point of the construction.

Lemma 6.2 *For each $\boldsymbol{F}(z) \in \mathcal{H}^s_-(z) \otimes \mathbb{C}^s$ satisfying the conditions 6.25 and 6.26, any $|v\rangle \in \mathcal{F}^s$ and any natural N we have the equality of elements of $\Lambda^{s,N}_-$:*

$$\mathrm{A}_N\, \pi_{N-1,1}(\boldsymbol{F}(z) \otimes |v\rangle) = \pi_N \mathcal{A}(\boldsymbol{F}(z))|v\rangle. \tag{6.27}$$

Proof. Let $\boldsymbol{F}(z)$ has the form

$$\boldsymbol{F}(z) = \sum_{c=1}^{s} F_c(z) \otimes e_c, \qquad F_c(z) \in \mathcal{H}^s_-(z)$$

Consider first the LHS of (6.27). This is the antisymetrization 6.23 of the tensor

$$\sum_{c_1,..,c_N=1}^{s} \langle 0|\Psi_{c_N}(x_N) \cdots \Psi_{c_2}(x_2) F_{c_1}(x_1)|v\rangle \cdot e_{c_1} \otimes \ldots \otimes e_{c_N}.$$

which can be written by means of proper changes of summation indices as the sum

$$\sum_{k=1}^{N} (-1)^{k+1} \sum_{c_1,..,c_N=1}^{s} \langle 0|\Psi_{c_N}(x_N) \cdots \Psi_{c_{k+1}}(x_{k+1}) \Psi_{c_{k-1}}(x_{k-1}) \cdots \Psi_{c_1}(x_1) F_{c_k}(x_k)|v\rangle \\ \cdot e_{c_1} \otimes \ldots \otimes e_{c_N}.$$

Using the relation

$$\int_{z \circlearrowleft x_k} dz\, \frac{F_{c_k}(z)}{x_k - z} = -F_{c_k}(x_k),$$

we rewrite the LHS of (6.27) as

$$\sum_{k=1}^{N} (-1)^{k} \sum_{c_1,..,c_N=1}^{s} \langle 0|\Psi_{c_N}(x_N) \cdots \Psi_{c_{k+1}}(x_{k+1}) \Psi_{c_{k-1}}(x_{k-1}) \cdots \Psi_{c_1}(x_1) \cdot \\ \frac{1}{2\pi i} \int_{z \circlearrowleft x_k} dz \frac{F_{c_k}(z)}{x_k - z} |v\rangle \cdot e_{c_1} \otimes \ldots \otimes e_{c_N}.$$

Using (6.14), we insert the integral:

$$\frac{-1}{2\pi i} \int_{u \circlearrowleft x_k} \Psi_{c_k}(x_k) \Psi^*_u(u) du = 1$$

into each summand of the k-th group. Then the LHS of (6.27) takes the form

$$
\begin{aligned}
&-\frac{1}{(2\pi i)^2}\sum_{k=1}^{N}\sum_{c_1,..,c_N=1}^{s}\int\limits_{z\circlearrowleft x_k} dz \int\limits_{\substack{u\circlearrowleft x_k\\ |z-x_k|\gg|u-x_k|}} du\ \langle 0|\prod_{N\ge i\ge 1}\Psi_{c_i}(x_i)\frac{\Psi^*_{c_k}(u)F_{c_k}(z)}{x_k-z}|v\rangle\cdot e_{c_1}\otimes\ldots\otimes e_{c_N}=\\
&-\frac{1}{(2\pi i)^2}\sum_{k=1}^{N}\sum_{c_1,..,c_N=1}^{s}\int\limits_{z\circlearrowleft x_k} dz \int\limits_{\substack{u\circlearrowleft x_k\\ |z-x_k|\gg|u-x_k|}} du\ \langle 0|\prod_{N\ge i\ge 1}\Psi_{c_i}(x_i)\frac{\Psi^*_{c_k}(u)F_{c_k}(z)}{u-z}|v\rangle\cdot e_{c_1}\otimes\ldots\otimes e_{c_N}.
\end{aligned}
\tag{6.28}
$$

Now in each summand we move the contour of integration for z close to the point x_k, crossing the singularity at $z=u$. Then the integral in every such summand transforms into the sum of two integrals,

$$
\begin{aligned}
&-\int\limits_{z\circlearrowleft x_k} dz \int\limits_{\substack{u\circlearrowleft x_k\\ |z-x_k|\gg|u-x_k|}} du\ \langle 0|\prod_{N\ge i\ge 1}\Psi_{c_i}(x_i)\frac{\Psi^*_{c_k}(u)F_{c_k}(z)}{u-z}|v\rangle=\\
&-\int\limits_{z\circlearrowleft x_k} dz \int\limits_{\substack{u\circlearrowleft x_k\\ |u-x_k|\gg|z-x_k|}} du\ \langle 0|\prod_{N\ge i\ge 1}\Psi_{c_i}(x_i)\frac{\Psi^*_{c_k}(u)F_{c_k}(z)}{u-z}|v\rangle+\\
&\int\limits_{z\circlearrowleft x_k} dz \int\limits_{u\circlearrowleft z} du\ \langle 0|\prod_{N\ge i\ge 1}\Psi_{c_i}(x_i)\frac{\Psi^*_{c_k}(u)F_{c_k}(z)}{u-z}|v\rangle
\end{aligned}
\tag{6.29}
$$

In the first integral, see the middle line of 6.29, after the change of the order of integration we observe its vanishing due to condition (i) of 6.25: there is no singularity of the integral at any point $z=x_j$. We now conclude that the LHS of 6.27 equals to the double integral

$$
\frac{1}{(2\pi i)^2}\sum_{k=1}^{N}\sum_{c_1,..,c_N=1}^{s}\int\limits_{z\circlearrowleft x_k} dz \int\limits_{u\circlearrowleft z} du\langle 0|\prod_{N\ge i\ge 1}\Psi_{c_i}(x_i)\frac{\Psi^*_{c_k}(u)F_{c_k}(z)}{u-z}|v\rangle\cdot e_{c_1}\otimes\ldots\otimes e_{c_N}
$$

or

$$
\frac{1}{(2\pi i)^2}\sum_{k=1}^{N}\sum_{c_1,..,c_N=1}^{s}\int\limits_{C} dz \int\limits_{u\circlearrowleft z} du\langle 0|\prod_{N\ge i\ge 1}\Psi_{c_i}(x_i)\frac{\Psi^*_{c_k}(u)|F_{c_k}(z)}{u-z}|v\rangle\cdot e_{c_1}\otimes\ldots\otimes e_{c_N},
\tag{6.30}
$$

where the contour C encloses all the points x_k but does not enclose zero.

On the other hand, the RHS of 6.27,

$$
\langle 0|\frac{1}{(2\pi i)^2}\prod_{N\ge i\ge 1}\boldsymbol{\Psi}(x_i)\oint dz\int_{u\circlearrowleft z} du\frac{\boldsymbol{\Psi}^*(u)\boldsymbol{F}(z)}{u-z}|v\rangle
$$

in components looks like

$$
\frac{1}{(2\pi i)^2}\sum_{k=1}^{N}\sum_{c_1,..,c_N=1}^{s}\langle 0|\prod_{N\ge i\ge 1}\Psi_{c_i}(x_i)\oint dz\int_{u\circlearrowleft z} du\ \frac{\Psi^*_{c_k}(u)F_{c_k}(z)}{u-z}|v\rangle\cdot e_{c_1}\otimes\ldots\otimes e_{c_N}.
$$

The region of analyticity of any matrix coefficient $\langle\xi|\prod_{N\geq i\geq 1}\Psi_{c_i}(x_i)\Psi^*_{c_k}(u)F(z)|v\rangle$ is $0<|x_N|<|x_{N-1}|<\ldots<|x_1|<|u|<|z|$ so the integral over z can be replaced by the contour integral over the contour enclosing all x_k and zero. Deforming this contour we see that the RHS of 6.27 equals to the sum 6.28 plus the integral which enclose zero and not the points x_k. To prove the equality 6.27 it is sufficient to verify that each integral

$$\int\limits_{z\circlearrowleft 0} dz \int\limits_{u\circlearrowleft z} du\ \langle 0|\prod_{N\geq i\geq 1}\Psi_{c_i}(x_i)\frac{\Psi^*_{c_k}(u)F_{c_k}(z)}{u-z}|v\rangle \tag{6.31}$$

vanishes. In the latter integral all singularities at the diagonals $z=x_i$ and $u=x_i$ are out of the domain of integration. Thus the vanishing of these integrals is equivalent to vanishing of the vector valued integral

$$\int\limits_{z\circlearrowleft 0} dz \int\limits_{u\circlearrowleft z} du\ \langle 0|\frac{\Psi^*_{c_k}(u)F_{c_k}(z)}{u-z}. \tag{6.32}$$

The domain of analyticity of the expression $\Psi^*_{c_k}(u)F_{c_k}(z)$ is $|u|<|z|$ and all the singularities are poles of finite order on the diagonal $z=u$, so that the relation

$$(z-u)^N\Psi^*_{c_k}(u)F_{c_k}(z)=(z-u)^N F_{c_k}(z)\Psi^*_{c_k}(u) \tag{6.33}$$

holds for sufficiently big N, where the sign is chosen according to parity of the field $F_{c_k}(z)$. The relation (6.33) implies that the analytic continuation of $\Psi^*_{c_k}(u)F_{c_k}(z)$ to the region $|u|>|z|$ is $\pm F_{c_k}(z)\Psi^*_{c_k}(u)$. By definition 6.4 of the vacuum state and the related rules of the normal ordering (6.5) the integral 6.32 can be formally rewritten as

$$\int_{z\circlearrowleft 0} dz\langle 0|\left(\Psi^*_{c_k,-}(z)F_{c_k}(z)\pm F_{c_k}(z)\Psi^*_{c_k,+}(z)\right), \tag{6.34}$$

where the sign depends on the parity of $F_{c_k}(z)=\sum_{n\in\mathbb{Z}}f_n z^n$ and

$$\Psi^*_{c_k,-}(z)=\sum_{n\leq 0}\psi^*_{c_k,n}z^{n-1},\qquad \Psi^*_{c_k,+}(z)=\sum_{n>0}\psi^*_{c_k,n}z^{n-1}.$$

In Fourier modes 6.32 looks as

$$\sum_{n\leq 0}\psi^*_{c_k,n}f_{-n}\pm\sum_{n>0}f_{-n}\psi^*_{c_k,n}.$$

The first sum vanishes due to 6.4. By assumption, $\deg F_{c_k}(z)=0$ thus $\deg f_n=-n$. We then see that in the second term all f_n have positive degree and if we assume them to be normal ordered they contain in each summand either ψ_{an} or ψ^*_{bn} with $n<0$ at their left end. Thus $\langle 0|f_{-n}=0$ for $n>0$ and the integral 6.32 vanishes. $\square$

Define an operator $\mathcal{D}:\mathcal{H}^s_-(z)\otimes\mathbb{C}^s\to\mathcal{H}^s_-(z)\otimes\mathbb{C}^s$ by the relation

$$\begin{aligned}&\mathcal{D}\boldsymbol{F}(z)=\alpha z\frac{d}{dz}\boldsymbol{F}(z)+\\ &\frac{z}{(2\pi i)^2}\int_{\substack{w\circlearrowleft 0\\ |w|<|z|}} dw\int_{u\circlearrowleft w} du\ \boldsymbol{\Psi}^{*(2)}(u)\frac{\boldsymbol{\Psi}^{(2)}(w)\boldsymbol{F}^{(1)}(z)-\boldsymbol{\Psi}^{(2)}(z)\boldsymbol{F}^{(1)}(w)}{(u-w)(z-w)}.\end{aligned} \tag{6.35}$$

Here upper indices (1) and (2) indicate tensor components where corresponding operators act. In components,

$$
\begin{aligned}
&\mathcal{D}F_c(z)\otimes e_c = \alpha z\frac{d}{dz}F_c(z)\otimes e_c +\\
&\frac{z}{(2\pi i)^2}\sum_{b=1}^{s}\int_{\substack{w\circlearrowleft 0\\ |w|<|z|}} dw \int_{u\circlearrowleft w} du\ \Psi_b^*(u)\frac{\Psi_b(w)F_c(z)-\Psi_b(z)F_c(w)}{(u-w)(z-w)}\otimes e_c.
\end{aligned}
$$

By means of Lemma 6.2 we now can identify the operator $\mathcal{D}$ as a pullback of the equivariant family of Heckman operators $\mathcal{D}_i^{(N)}$ acting in the space of partially antisymmetric tensors

Proposition 6.1 *For any $\boldsymbol{F}(z)\in\mathcal{H}_-^s(z)\otimes\mathbb{C}^s$ satisfying the condition 6.25 and 6.26, $|v\rangle\in\mathcal{F}^s$ and $N\in\mathbb{N}$ we have the equality*

$$
\pi_{N-1,1}(\mathcal{D}\boldsymbol{F}(x_1)\otimes|v\rangle)=\mathcal{D}_1^{(N)}\pi_{N-1,1}(F(x_1)\otimes|v\rangle). \tag{6.36}
$$

Proof. First we note that once the element $\boldsymbol{F}(z)\in\mathcal{H}_-^s(z)\otimes\mathbb{C}^s$ satisfies the conditions 6.25 and 6.26, the same is true for the divided difference

$$
\frac{\boldsymbol{\Psi}^{(2)}(w)\boldsymbol{F}^{(1)}(z)-\boldsymbol{\Psi}^{(2)}(z)\boldsymbol{F}^{(1)}(w)}{z-w}.
$$

The property 6.25 is valid because both the differential and difference derivatives preserve the polynomial property. The property 6.26 is evident: the difference derivatives are homogeneous of degree zero. We thus can use Lemma 6.2. Now the rest of the proof is identical to the proof of Proposition 5.1. □

Note that the application of the operator $\mathcal{D}$ to some $\boldsymbol{F}(z)\in\mathcal{H}_-^s(z)\otimes\mathbb{C}^s$, which satisfies the conditions 6.25 and 6.26, preserves these conditions by the same reasons of homogeneity and preservation of polynomial spaces by both difference and differential derivatives. This gives rise to the formulas for pullback of sum of powers of Dunkl operators.

Let $E_{ab}\in\operatorname{End}\mathbb{C}^s$, be the matrix unit, $E_{ab}(e_c)=\delta_{bc}e_a$. Denote by $\mathcal{E}_{ab}$, the operator $1\otimes E_{ab}:\mathcal{H}_-^s(z)\otimes\mathbb{C}^s\to\mathcal{H}_-^s(z)\otimes\mathbb{C}^s$:

$$
\mathcal{E}_{ab}\boldsymbol{F}(z)=F_b(z)\otimes e_a.
$$

For $a,b=1,...,s$ and $n=1,...$ define the element $\mathrm{T}_{ab,n}\in\mathcal{H}_-^s$ by the relation

$$
\mathrm{T}_{ab,n}=\mathcal{A}\mathcal{E}_{ab}\mathcal{D}^n\boldsymbol{\Psi}(z)=\frac{1}{2\pi i}\int_{z\circlearrowleft 0} dz\ \mathcal{T}_{ab,n}. \tag{6.37}
$$

Here $\mathcal{T}_{ab,n}$ is the n-th order density defined by the formula:

$$
\mathcal{T}_{ab,n}=\frac{1}{2\pi i}\int_{u\circlearrowleft z} du\frac{\boldsymbol{\Psi}^*(u)\mathcal{E}_{ab}\mathcal{D}^n\boldsymbol{\Psi}(z)}{u-z}. \tag{6.38}
$$

In Appendix we give expressions for the first densities for $n=0,1,2$ in terms of normal ordering fermionic fields and in terms of generators of the affine Lie algebra $\widehat{\mathfrak{gl}}_s$, see (6.15).

Summarizing the statements above we establish the operator $\mathrm{T}_{ab,n}$ as the pullback of the Yangian generator $t_{ab,n}$ in $\Lambda_-^{s,N}$.

Proposition 6.2 *For any $|v\rangle \in \mathcal{F}^s$ and $N \in \mathbb{N}$ we have the equality*

$$\pi_N(\mathrm{T}_{ab,n}|v\rangle) = t_{ab,n}\pi_N|v\rangle.$$

We now reformulate projective properties of the Yangian action in the phase space of finite-dimensional CS system, see Proposition 4.2, in terms of the constructed operators in the Fock space.

Lemma 6.3 *For any $|v\rangle \in \mathcal{F}^s$ we have the equality*

$$\pi_N(Q|v\rangle) = \omega^-_{N+s} \cdot \pi_{N+s}(|v\rangle). \tag{6.39}$$

Proof. The LHS of 6.39 reads

$$\begin{aligned}\pi_N(Q|v\rangle) =& \langle 0|\mathbf{\Psi}(x_N)\cdots\mathbf{\Psi}(x_2)\mathbf{\Psi}(x_1)Q|v\rangle = \\ & (x_1\cdots x_N)^{-1}\cdot\langle 0|Q\mathbf{\Psi}(x_N)\cdots\mathbf{\Psi}(x_2)\mathbf{\Psi}(x_1)|v\rangle = \\ & (x_1\cdots x_N)^{-1}\cdot\langle 0|\psi_{s,0}\cdots\psi_{1,0}\mathbf{\Psi}(x_N)\cdots\mathbf{\Psi}(x_2)\mathbf{\Psi}(x_1)|v\rangle.\end{aligned}$$

The last line is precisely the RHS of 6.39. Indeed,

$$\omega^-_{N+s}\cdot\pi_{N+s}(|v\rangle) = \omega^-_{N+s}\cdot\sum_{c_1,..,c_{N+s}=1}^{s}\langle 0|\Psi_{c_{N+s}}(x_{N+s})\cdots\Psi_{c_2}(x_2)\Psi_{c_1}(x_1)|v\rangle\cdot e_{c_1}\otimes\ldots\otimes e_{c_{N+s}}$$

$$= (x_1\cdots x_N)^{-1}\cdot\sum_{c_1,..,c_N=1}^{s}\langle 0|\psi_{s,0}\cdots\psi_{1,0}\Psi_{c_N}(x_N)\cdots\Psi_{c_2}(x_2)\Psi_{c_1}(x_1)|v\rangle\cdot e_{c_1}\otimes\ldots\otimes e_{c_N}.$$

□

Denote by $\mathrm{T}(u)$ the generating matrix of operators $\mathrm{T}_{ab,n}$

$$\mathrm{T}(u) = \sum_{a,b=1}^{s} E_{ab}\otimes\mathrm{T}_{ab}(u) \in \mathrm{End}(\mathbb{C}^s)\otimes\mathcal{H}^s_-[u^{-1}],$$

where

$$\mathrm{T}_{ab}(u) = \delta_{ab} + \sum_{n\geq 0}\mathrm{T}_{ab,n}\,u^{-n}.$$

Proposition 4.2 and Lemma 6.3 imply

Corollary 6.1 *The following identity holds*

$$Q\,\mathrm{T}(u)Q^{-1} = \frac{u+1}{u}\,\mathrm{T}(u-\alpha-s) \tag{6.40}$$

The equality 6.40 can be regarded as a recurrence relations which expresses each $\mathrm{T}_{ab,n}$ via $\mathrm{Ad}_Q(\mathrm{T}_{ab,k})$ for $k<n$. In particular, this means that

$$(\mathrm{Ad}_Q - 1)^{n+2}(\mathrm{T}_{ab,n}) = 0 \tag{6.41}$$

and thus any operator $A = \mathrm{T}_{ab,n}$ can be presented as a polynomial of degree $n+1$

$$A = A_0 + a_0A_1 + \ldots + a_0^{n+1}A_{n+1}, \qquad QA_jQ^{-1} = A_j, \tag{6.42}$$

where each A_j is an element of $\mathcal{H}$ of zero charge. In its turn, the presentation (6.42) implies

Theorem 6.1 *The operators $\mathrm{T}_{ab,n}$ satisfy Yangian relations (4.3).*

In particular, the coefficients of the quantum determinant $q\det \mathrm{T}(u)$ form a commutative family which can be regarded as the limits of the higher Hamiltonians of CS system. Indeed, due 6.42, each Yangian relation is polynomial over a_0 and thus it is enough to verify it on subspaces $\mathcal{F}^s_N$, which are eigenspaces of a_0 with eigenvalue N for N big enough. But here in the projection π_N we deal with polynomials of any desirable degree where the relation becomes nontrivial.

Analogously to finite-dimensional case, see 4.30, the transfer matrix $\mathrm{T}(u)$ can be renormalized by means of central operator a_0 in such a way that the new transfer matrix will commute with Q and thus acts in equal way in each sector $\mathcal{F}^s_N$ of the Fock space. Here we set

$$\bar{\mathrm{T}}(u) = f(u, a_0)\,\mathrm{T}\left(u + a_0\frac{\gamma}{s}\right), \tag{6.43}$$

where $\gamma = \alpha + s$ and

$$f(u, b) = \frac{\Gamma\left(\frac{u}{\gamma} + \frac{b}{s} + 1\right)\Gamma\left(\frac{u+1}{\gamma} + 1\right)}{\Gamma\left(\frac{u+1}{\gamma} + \frac{b}{s} + 1\right)\Gamma\left(\frac{u}{\gamma} + 1\right)}. \tag{6.44}$$

Comments. Denis Uglov presented two construction of the Yangian action in fermionic space $\mathcal{F}^s$, starting from Yangian action in the phase space of fermionic Spin CS system. The paper [57] deals with projective type construction while the work [58] develops an inductive limit approach. Our paper has a close connection with [57].

Initial data of the present work and that of [57] are the same: finite-dimensional representations of Yangians realized via polynomial representations of degenerated affine Hecke algebra. In this setting D. Uglov uses projective properties of these action and renormalize the transfer matrices of Yangian action in order to form the projective system. He identifies zero charge subspace of $\mathcal{F}^s$ with the projective limit of $\Lambda_-^{s,n}$ and defines the Yangian action on $\mathcal{F}^s_0$ via this identification. The action is extended to other sectors by means of natural identifications of these sectors with $\mathcal{F}^s_0$. The resulting Yangian action is given by implicit formula, analogous (but not the same!) to 4.34, and actually coincides with renormalized finite-dimensional action on stable wedge. Using this description and representation theory of degenerate affine Hecke algebra, D. Uglov suggested precise decomposition of the Fock space into direct sum of Yangian irreducibles.

Our construction can be regarded as a free field counterpart of Uglov investigations. However, there are certain differences in two approaches. First, we use different projections from the Fock space to spin CS phase space $\Lambda_-^{s,N}$. They differ by the power of Q. Namely, Uglov projection $\tilde{\pi}_{Ns} : \mathcal{F}^s_0 \to \Lambda_-^{s,Ns}$ can be expressed via 6.21 by the relation

$$\tilde{\pi}_{Ns} = \pi_{Ns} Q^{-N}.$$

The use of our projections allows to lift the initial action to the whole Fock space without renormalization. However, after renormalization 6.43 both actions should coincide.

Second, in his normalization procedure [57, Proposition 10.2] Uglov lost the shift of spectral parameter which led to disagreement in final results. This disagreement does not affect the decomposition into Yangian irreducibles, but changes the parameters of irreducibles. Namely, one should twist Uglov irreducible components by certain automorphisms of the Yangian. Surely, after the mentioned changes Uglov decomposition can be equally used in our interpretation of the model.

Conclusion

In this work the limits of the Calogero-Sutherland system when the number of particles N tends to infinity were investigated. We studied the bosonic and fermionic limit corresponding to the symmetric and antisymmetric wave functions of the system.

For the fermionic limit of the scalar system, we derived a limit expression for the Dunkl operator via free fermionic fields, which allows us to present the construction of commuting Hamiltonians in the Fock space. In the case of the value of the coupling constant $\beta = 0$, we have presented an explicit formula for the generating function of Hamiltonians that differs from the previously known ones. This result may find some applications in such areas as knot theory and combinatorics of Hurwitz numbers. In case of arbitrary value of coupling constant to find the precise generating function of Hamiltonians is an open problem.

For spin system we realized the bosonic and fermionic limit in a multicomponent Fock space. We introduced the maps to finite system and construct the pullback of finite Dunkl operators in terms of vertex operators in bosonic case and in terms of free fermion fields in fermionic case. We constructed the corresponding Yangian representation in the Fock space, which may have nontrivial applications in the representation theory.

Appendix

Here we present the expressions for the first densities $\mathcal{T}_{ab,n}(z)$ (6.38) $n = 0, 1, 2$ for the Yangian generators. There will be given two types of expressions for each density, the first answer is a normal ordered combination of fermionic fields $\Psi_c(z)$, $\Psi^*_d(z)$, the second is not normal ordered, it is given in terms the affine Lie algebra $\widehat{\mathfrak{gl}}_s$ generators.

Now we introduce several notations. Denote by $\mathcal{T}^{kl}_{ab}(z)$ a coefficient of α^l in $\mathcal{T}_{ab,k+l}(z)$:

$$\mathcal{T}_{ab,n}(z) = \sum_{l=0}^{n} \alpha^l \mathcal{T}^{n-l,l}_{ab}(z).$$

Denote by $E_{ab}(z)$ a generating functions for the elements of the affine Lie algebra $\widehat{\mathfrak{gl}}_s$:

$$E_{ab}(z) = \sum_n E_{ab,n} z^n = \vdots z\Psi^*_a(z)\Psi_b(z)\vdots.$$

For a formal series $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$ we denote by $f(z)_\pm$ the series

$$f(z)_+ = \sum_{n\geq 0} f_n z^n = \int\limits_{\substack{u\circlearrowleft 0\\ |u|\gg|z|}} du\, \frac{f(u)}{u-z}, \qquad f(z)_- = \sum_{n<0} f_n z^n = \int\limits_{\substack{u\circlearrowleft 0\\ |u|\ll|z|}} du\, \frac{f(u)}{z-u}.$$

For $n = 0$ we simply have

$$\mathcal{T}_{ab,0}(z) = \mathcal{T}^{0,0}_{ab}(z) = \vdots\Psi^*_a(z)\Psi_b(z)\vdots = \frac{1}{z}E_{ab}(z). \tag{6.1}$$

For $n = 1$

$$\mathcal{T}_{ab,1}(z) = \alpha\mathcal{T}^{0,1}_{ab}(z) + \mathcal{T}^{1,0}_{ab}(z).$$

We distinguish the answers for diagonal $\mathcal{T}_{aa,n}(z)$ and nondiagonal part $\mathcal{T}_{ab,n}(z)$, where $a \neq b$. Firstly we present the expressions for nondiagonal elements $a \neq b$ as normal ordered combination of fermionic fields:

$$\mathcal{T}^{0,1}_{ab}(z) = \vdots\Psi^*_a(z) z\frac{\partial}{\partial z}\Psi_b(z)\vdots,$$

$$\begin{aligned}\mathcal{T}^{1,0}_{ab}(z) =& \vdots \sum_{c=1}^{s} z\Psi^*_a(z)\Psi_b(z)\left(\Psi^*_c(z)\Psi_c(z)\right)_- + \sum_{c=1}^{s} z\Psi^*_a(z)\Psi_c(z)\left(\Psi^*_c(z)\Psi_b(z)\right)_- \\ &+ (s+1)\ \Psi^*_a(z)\left(z\frac{\partial}{\partial z}\Psi_b(z)\right)_- - \Psi_b(z)\left(z\frac{\partial}{\partial z}\Psi^*_a(z)\right)_+ \vdots.\end{aligned}$$

The bosonic answer has the recurrent form, we express it from $\mathcal{T}^{0,0}_{ab}(z)$ (6.1):

$$\mathcal{T}^{0,1}_{ab}(z) = \int\limits_{w\circlearrowleft z} \frac{dw}{(w-z)}\ E_{aa}(z)\mathcal{T}^{0,0}_{ab}(w),$$

$$\mathcal{T}^{1,0}_{ab}(z)=\sum_{c=1}^{s}\int\limits_{\substack{w\circlearrowleft 0\\|w|\ll|z|}}\frac{dw}{(z-w)}E_{ac}(z)\mathcal{T}^{0,0}_{cb}(w)+\sum_{c=1}^{s}\int\limits_{\substack{w\circlearrowleft 0\\|w|\ll|z|}}\frac{zdw}{w(z-w)}\mathcal{T}^{0,0}_{ab}(z)E_{cc}(w)-$$
$$-\int\limits_{w\circlearrowleft z}\frac{zdw}{w(w-z)}\mathcal{T}^{0,0}_{ab}(z)E_{aa}(w).$$

For diagonal elements in case $n=1$ we present the answers in the same way, firstly as a normal ordered combination of fermionic fields:

$$\mathcal{T}^{0,1}_{aa}(z)=\vdots\Psi^*_a(z)z\frac{\partial}{\partial z}\Psi_a(z)\vdots,$$

$$\mathcal{T}^{1,0}_{aa}(z)=\vdots\sum_{b=1}^{s}z\Psi^*_a(z)\Psi_a(z)\left(\Psi^*_b(z)\Psi_b(z)\right)_-+\sum_{b=1}^{s}z\Psi^*_a(z)\Psi_b(z)\left(\Psi^*_b(z)\Psi_a(z)\right)_-$$
$$-\sum_{b=1}^{s}\Psi_b(z)\left(z\frac{\partial}{\partial z}\Psi^*_b(z)\right)_++(s+1)\ \Psi^*_a(z)\left(z\frac{\partial}{\partial z}\Psi_a(z)\right)_--\Psi_a(z)\left(z\frac{\partial}{\partial z}\Psi^*_a(z)\right)_+\vdots.$$

Then the recurrent answer from previous densities in terms the affine Lie algebra $\widehat{\mathfrak{gl}}_s$ generators:

$$\mathcal{T}^{0,1}_{aa}(z)=\frac{1}{2}\int\limits_{w\circlearrowleft z}\frac{dw}{(w-z)}\ E_{aa}(z)\mathcal{T}^{0,0}_{aa}(w)+\frac{1}{2}\int\limits_{w\circlearrowleft z}\frac{zdw}{(w-z)^2}\mathcal{T}^{0,0}_{aa}(w),$$

$$\mathcal{T}^{1,0}_{aa}(z)=\sum_{c=1}^{s}\int\limits_{\substack{w\circlearrowleft 0\\|w|\ll|z|}}\frac{dw}{(z-w)}E_{ac}(z)\mathcal{T}^{0,0}_{ca}(w)+\sum_{c=1}^{s}\int\limits_{\substack{w\circlearrowleft 0\\|w|\ll|z|}}\frac{zdw}{w(z-w)}\mathcal{T}^{0,0}_{aa}(z)E_{cc}(w)-$$
$$-\sum_{c=1}^{s}\left(\int\limits_{w\circlearrowleft z}\frac{dw}{(w-z)}\ E_{cc}(z)\mathcal{T}^{0,0}_{cc}(w)-\mathcal{T}^{0,1}_{cc}(z)\right)+\int\limits_{w\circlearrowleft z}\frac{zdw}{(w-z)^2}\mathcal{T}^{0,0}_{aa}(w)-T^{0,1}_{aa}(z).$$

For $n=2$

$$\mathcal{T}_{ab,2}(z)=\alpha^2\mathcal{T}^{0,2}_{ab}(z)+\alpha\mathcal{T}^{1,1}_{ab}(z)+\mathcal{T}^{2,0}_{ab}(z)$$

We split $\mathcal{T}^{1,1}_{ab}(z)$ into two summands :

$$\mathcal{T}^{1,1}_{ab}(z)=(\mathcal{T}^{1,1}_{ab})'(z)+(\mathcal{T}^{1,1}_{ab})''(z).$$

Here $(\mathcal{T}^{1,1}_{ab})'(z)$ means that firstly we apply $z\frac{\partial}{\partial z}$ and then the difference part of the Dunkl operator, $(\mathcal{T}^{1,1}_{ab})''(z)$ backwards.

$$\mathcal{T}^{0,2}_{ab}(z)=\vdots\Psi^*_a(z)\left(z\frac{\partial}{\partial z}\right)^2\Psi_b(z)\vdots$$

$$(\mathcal{T}^{1,1}_{ab})'(z)=\vdots\sum_{c=1}^{s}z\Psi^*_a(z)\left(z\frac{\partial}{\partial z}\Psi_b(z)\right)\left(\Psi^*_c(z)\Psi_c(z)\right)_-+\sum_{c=1}^{s}z\Psi^*_a(z)\Psi_c(z)\left(\Psi^*_c(z)z\frac{\partial}{\partial z}\Psi_b(z)\right)_-$$
$$+\left(s+\frac{1}{2}\right)\ \Psi^*_a(z)\left(\left(z\frac{\partial}{\partial z}\right)^2\Psi_b(z)\right)_--\left(z\frac{\partial}{\partial z}\Psi_b(z)\right)\left(z\frac{\partial}{\partial z}\Psi^*_a(z)\right)_+-\frac{1}{2}\Psi^*_a(z)\left(z\frac{\partial}{\partial z}\Psi_b(z)\right)_-\vdots$$

$$(\mathcal{T}^{1,1}_{ab})''(z) = \vdots \sum_{c=1}^{s} \Psi^*_a(z) z\frac{\partial}{\partial z}\left(z\Psi_c(z)\left(\Psi^*_c(z)\Psi_b(z)\right)_-\right) + \sum_{c=1}^{s} \Psi^*_a(z) z\frac{\partial}{\partial z}\left(z\Psi_b(z)\left(\Psi^*_c(z)\Psi_c(z)\right)_-\right)$$
$$+ (s+1)\ \Psi^*_a(z)\left(\left(z\frac{\partial}{\partial z}\right)^2\Psi_b(z)\right)_- - \frac{1}{2}\Psi_b(z)\left(z\frac{\partial}{\partial z}\Psi^*_a(z)\right)_+$$
$$-\left(z\frac{\partial}{\partial z}\Psi_b(z)\right)\left(z\frac{\partial}{\partial z}\Psi^*_a(z)\right)_+ - \frac{1}{2}\Psi_b(z)\left(\left(z\frac{\partial}{\partial z}\right)^2\Psi^*_a(z)\right)_+ \vdots$$

The recurrent formula from previous densities in terms the affine Lie algebra $\widehat{\mathfrak{gl}}_s$ generators:

$$\mathcal{T}^{0,2}_{ab}(z) = \int\limits_{w\circlearrowleft z} \frac{dw}{(w-z)}\ E_{aa}(z)\mathcal{T}^{0,1}_{ab}(w)$$

$$(\mathcal{T}^{1,1}_{ab})'(z) = \sum_{c=1}^{s} \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{dw}{(z-w)} E_{ac}(z)\mathcal{T}^{0,1}_{cb}(w) + \sum_{c=1}^{s} \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{zdw}{w(z-w)} \mathcal{T}^{0,1}_{ab}(z)E_{cc}(w) -$$
$$- \int\limits_{w\circlearrowleft z} \frac{zdw}{w(w-z)} \mathcal{T}^{0,1}_{ab}(z)E_{aa}(w)$$

$$(\mathcal{T}^{1,1}_{ab})''(z) = \int\limits_{w\circlearrowleft z} \frac{dw}{(w-z)} E_{aa}(z)\mathcal{T}^{1,0}_{ab}(w) + \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{zdw}{w(z-w)} \left(\mathcal{T}^{0,0}_{aa}(z) + z\frac{\partial}{\partial z}\mathcal{T}^{0,0}_{aa}(z) - \mathcal{T}^{0,1}_{aa}(z)\right) E_{ab}(w) -$$
$$- \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{zdw}{w(z-w)^2} E_{aa}(z)E_{ab}(w) - \int\limits_{w\circlearrowleft z} \frac{zdw}{w(w-z)} \mathcal{T}^{0,1}_{ab}(z)E_{aa}(w)$$
$$+ \int\limits_{w\circlearrowleft z} \frac{zdw}{(w-z)^3} E_{ab}(w) - \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{zdw}{(z-w)^3} E_{ab}(w) - \frac{1}{2}\mathcal{T}^{0,2}_{ab}(z) - \frac{1}{2}\mathcal{T}^{0,1}_{ab}(z)$$

For diagonal elements in case $n=2$ we have more complicated formulas:

$$\mathcal{T}^{0,2}_{aa}(z) = \vdots \Psi^*_a(z)\left(z\frac{\partial}{\partial z}\right)^2\Psi_a(z)\vdots$$

$$(\mathcal{T}^{1,1}_{aa})'(z) = \vdots \sum_{c=1}^{s} z\Psi^*_a(z)\left(z\frac{\partial}{\partial z}\Psi_a(z)\right)\left(\Psi^*_c(z)\Psi_c(z)\right)_- + \sum_{c=1}^{s} z\Psi^*_a(z)\Psi_c(z)\left(\Psi^*_c(z)z\frac{\partial}{\partial z}\Psi_a(z)\right)_-$$
$$+ \left(s+\frac{1}{2}\right)\ \Psi^*_a(z)\left(\left(z\frac{\partial}{\partial z}\right)^2\Psi_a(z)\right)_- + \frac{1}{2}\sum_{c=1}^{s}\Psi_c(z)\left(\left(z\frac{\partial}{\partial z}\right)^2\Psi^*_c(z)\right)_+$$
$$+ \frac{1}{2}\sum_{c=1}^{s}\Psi_c(z)\left(z\frac{\partial}{\partial z}\Psi^*_c(z)\right)_+ - \left(z\frac{\partial}{\partial z}\Psi_a(z)\right)\left(z\frac{\partial}{\partial z}\Psi^*_a(z)\right)_+ \vdots$$

$$
\begin{aligned}
(\mathcal{T}^{1,1}_{aa})''(z) =\vdots & \sum_{b=1}^{s} \Psi^*_a(z) z\frac{\partial}{\partial z}\left(z\Psi_a(z)\left(\Psi^*_b(z)\Psi_b(z)\right)_-\right) + \sum_{b=1}^{s} \Psi^*_a(z) z\frac{\partial}{\partial z}\left(z\Psi_b(z)\left(\Psi^*_b(z)\Psi_a(z)\right)_-\right) \\
& - \frac{1}{2}\sum_{b=1}^{s} \Psi_b(z)\left(z\frac{\partial}{\partial z}\Psi^*_b(z)\right)_+ - \frac{1}{2}\sum_{b=1}^{s} \Psi_b(z)\left(\left(z\frac{\partial}{\partial z}\right)^2 \Psi^*_b(z)\right)_+ \\
& - \sum_{b=1}^{s}\left(z\frac{\partial}{\partial z}\Psi_b(z)\right)\left(z\frac{\partial}{\partial z}\Psi^*_b(z)\right)_+ + (s+1)\ \Psi^*_a(z)\left(\left(z\frac{\partial}{\partial z}\right)^2 \Psi_a(z)\right)_- \\
& - \frac{1}{2}\Psi_a(z)\left(z\frac{\partial}{\partial z}\Psi^*_a(z)\right)_+ - \frac{1}{2}\Psi_a(z)\left(\left(z\frac{\partial}{\partial z}\right)^2 \Psi^*_a(z)\right)_+ - \left(z\frac{\partial}{\partial z}\Psi_a(z)\right)\left(z\frac{\partial}{\partial z}\Psi^*_a(z)\right)_+ \vdots
\end{aligned}
$$

$$
\mathcal{T}^{0,2}_{aa}(z) = \frac{2}{3}\int\limits_{w\circlearrowleft z} \frac{dw}{(w-z)}\ E_{aa}(z)\mathcal{T}^{0,1}_{aa}(w) - \frac{2}{3}\int\limits_{w\circlearrowleft z} \frac{zdw}{(w-z)^3} E_{aa}(w) + \frac{2}{3}\int\limits_{w\circlearrowleft z} \frac{zdw}{(w-z)^2}\mathcal{T}^{0,1}_{aa}(w) + \frac{1}{3}\mathcal{T}^{0,1}_{aa}(z)
$$

$$
\begin{aligned}
(\mathcal{T}^{1,1}_{aa})'(z) = & \sum_{c=1}^{s} \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{dw}{(z-w)} E_{ac}(z)\mathcal{T}^{0,1}_{ca}(w) + \sum_{c=1}^{s} \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{zdw}{w(z-w)}\mathcal{T}^{0,1}_{aa}(z)E_{cc}(w) - \\
& - \sum_{c=1}^{s}\left(\int\limits_{w\circlearrowleft z} \frac{dw}{(w-z)}\ E_{cc}(z)\mathcal{T}^{0,1}_{cc}(w) - \mathcal{T}^{0,2}_{cc}(z)\right) + \int\limits_{w\circlearrowleft z} \frac{zdw}{(w-z)^2}\mathcal{T}^{0,1}_{aa}(w) - T^{0,2}_{aa}(z)
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{T}^{1,1}_{aa})''(z) = & \int\limits_{w\circlearrowleft z} \frac{dw}{(w-z)} E_{aa}(z)\mathcal{T}^{1,0}_{aa}(w) + \\
& + \sum_{b=1}^{s} \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{zdw}{w(z-w)}\left(\mathcal{T}^{0,0}_{aa}(z) + z\frac{\partial}{\partial z}\mathcal{T}^{0,0}_{aa}(z) - \mathcal{T}^{0,1}_{aa}(z)\right) E_{bb}(w) - \\
& - \sum_{b=1}^{s} \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{zdw}{w(z-w)^2} E_{aa}(z)E_{bb}(w) - \sum_{b=1}^{s} \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{zdw}{(z-w)^3} E_{bb}(w) \\
& + \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{zdw}{w(z-w)}\left(\mathcal{T}^{0,0}_{aa}(z) + z\frac{\partial}{\partial z}\mathcal{T}^{0,0}_{aa}(z) - \mathcal{T}^{0,1}_{aa}(z)\right) E_{aa}(w) \\
& - \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{zdw}{w(z-w)^2} E_{aa}(z)E_{aa}(w) - \int\limits_{\substack{w\circlearrowleft 0\\ |w|\ll|z|}} \frac{zdw}{(z-w)^3} E_{aa}(w) \\
& + \sum_{b=1}^{s} \int\limits_{w\circlearrowleft z} \frac{dw}{(w-z)}\left(\mathcal{T}^{0,1}_{aa}(z) - z\frac{\partial}{\partial z}\mathcal{T}^{0,0}_{aa}(z)\right) E_{bb}(w)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{b=1}^{s}\int\limits_{w\circlearrowleft z}\frac{zdw}{(w-z)^3}E_{bb}(w)-\sum_{b=1}^{s}\int\limits_{w\circlearrowleft z}\frac{dw}{(w-z)^2}E_{aa}(z)E_{bb}(w)\\
&+s\left(\int\limits_{w\circlearrowleft z}\frac{zdw}{(w-z)^2}\mathcal{T}_{aa}^{0,1}(w)-\int\limits_{w\circlearrowleft z}\frac{zdw}{(w-z)^3}E_{aa}(w)\right)-\frac{s}{2}\mathcal{T}_{aa}^{0,2}(z)+\frac{s}{2}\mathcal{T}_{aa}^{0,1}(z)\\
&+2\int\limits_{w\circlearrowleft z}\frac{zdw}{(w-z)^3}E_{aa}(w)-2\int\limits_{w\circlearrowleft z}\frac{zdw}{(w-z)^2}\mathcal{T}_{aa}^{0,1}(w)+\int\limits_{w\circlearrowleft z}\frac{zdw}{w(w-z)^2}E_{aa}(w)+\mathcal{T}_{aa}^{0,2}(z).
\end{aligned}
$$

The density $\mathcal{T}_{11}^{2,0}(z)$ has a cumbersome form and we do not present it here. In scalar case ($s=1$) the matrix coefficient $T_{11}^{2,0}$ is a polynomial in zero mode of the scalar bosonic field:

$$
T_{11}^{2,0}=\frac{1}{6}\left(2a_0^3-3a_0^2+a_0\right).
$$

In scalar case the same is for higher orders: the matrix coefficient $T_{11}^{n,0}$ is a polynomial of degree $(n+1)$ in zero mode of the scalar bosonic field [35].

References

[1] A. G. Abanov, P. B. Wiegmann, Quantum hydrodynamics, the quantum Benjamin-Ono equation, and the Calogero model, Physical review letters, 95(7), (2005), 076402

[2] M. J. Ablowitz, A. S. Fokas, J. Satsuma, H. Segur,On the periodic intermediate long wave equation, Journal of Physics A: Mathematical and General, 15(3), (1982), 781.

[3] A. Alexandrov, A. Zabrodin Free fermions and tau-functions, Journal of Geometry and Physics 67 (2013): 37-80.

[4] I. Andric, A. Jevicki and H. Levine, On the large-N limit in symplectic matrix models, Nucl. Phys. **B215** (1983), 307.

[5] T. Arakawa, Drinfeld functor and finite-dimensional representations of the Yangian, Commun. Math. Phys. **205** (1999), 1–18.

[6] H. Awata, Y. Matsuo, S. Odake, J. Shiraishi, Collective field theory, Calogero-Sutherland model and generalized matrix models, Physics Letters **B 347**:1 (1995), 49-55.

[7] H. Awata, Y. Matsuo and T. Yamamoto, Collective field description of spin Calogero-Sutherland models, J. Phys. **A29** (1996), 3089-3098.

[8] T. B. Benjamin, Internal waves of permanent form in fluids of great depth, Journal of Fluid Mechanics, 29(3), (1967), 559-592.

[9] D. Bernard, M. Gaudin, F. D. M. Haldane, V. Pasquier, Yang-Baxter equation in spin chains with long range interactions, J. Phys. **A26** (1993), 5219.

[10] F. Calogero, Solution of the one-dimensional n-body problems with quadratic and/or inversely quadratic pair potentials, Journal of Mathematical Physics, 12(3), (1971), 419-436,
F. Calogero, Solution of a three-body problem in one dimension, Journal of Mathematical Physics, 10(12), (1969), 2191-2196.
F. Calogero, Exactly solvable one-dimensional many-body problems, Lettere al Nuovo Cimento (1971-1985), 13(11), (1975), 411-416.

[11] I. Cherednik, A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras, Inventiones mathematicae 106.1 (1991): 411-431.

[12] I. Cherednik, Double affine Hecke algebras and Macdonald's conjectures. Annals of mathematics, 141(1), (1995), 191-216.

[13] A. Debiard, Polynômes de Tchébychev et de Jacobi dans un espace euclidien de dimension p, CR Acad. Sc. Paris, 296, (1983) 529-532.

[14] V. G. Drinfeld, Degenerate affine Hecke algebras and Yangians, Funct. Anal. Appl., **20**:1 (1986) 62–64.

[15] C. F. Dunkl, Differential-difference operators associated to reflection groups, Transactions of the American Mathematical Society. **311**:1 (1989), 167-183.

[16] A. S. Fokas, B. Fuchssteiner, The hierarchy of the Benjamin-Ono equation, Physics letters A, 86(6-7), (1981), 341-345.

[17] G. J. Heckman, An elementary approach to the hypergeometric shift operators of Opdam, Inventiones mathematicae **103**:1 (1991), 341-350.

[18] G. J. Heckman, A Remark on the Dunkl Differential—Difference Operators, In Harmonic analysis on reductive groups. Birkhäuser, Boston, MA. (1991), pp. 181-191.

[19] G. J. Heckman, E. M. Opdam, Root systems and hypergeometric functions. I. Compositio Mathematica, 64(3), (1987), 329-352.

[20] G. J. Heckman, Root systems and hypergeometric functions. II., Compositio mathematica, 64(3), (1987), 353-373.

[21] H. Jack, I.—A class of symmetric polynomials with a parameter, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 69:1, (1970), 1-18.

[22] A. Jevicki, B. Sakita. Collective field approach to the large-N limit: Euclidean field theories, Nuclear Physics B 185.1 (1981): 89-100.

[23] Y. Kato and Y. Kuramoto, Exact solution of the Sutherland model with arbitrary internal symmetry Phys. Rev. Lett. **74** (1995), 1222.

[24] D.J. Kaup, T.I. Lakoba, Y. Matsuno, Complete integrability of the Benjamin-Ono equation by means of action-angle variables, Physics Letters A, 238(2-3), (1998), 123-133.

[25] S.M. Khoroshkin, M.G. Matushko, Fermionic limit of the Calogero-Sutherland system, Journal of Mathematical Physics 60, (2019).

[26] S.M. Khoroshkin, M.G. Matushko, Matrix elements of vertex operators and fermionic limit of spin Calogero–Sutherland system, Journal of Physics A: Mathematical and Theoretical, arXiv:1608.00599.

[27] S.M. Khoroshkin, M.G. Matushko, E.K.Sklyanin, On spin Calogero–Moser system at infinity, Journal of Physics A: Mathematical and Theoretical, **50**:11 (2017), 115203.

[28] S. M. Khoroshkin, M. L. Nazarov, Yangians and Mickelsson algebras. II, Moscow Mathematical Journal **6**:3 (2006), 477-504.

[29] I. Krichever, O. Babelon, E. Billey, M. Talon, Spin generalization of the Calogero-Moser system and the matrix KP equation, Translations of the American Mathematical Society-Series 2, 170, (1995), 83-120.

[30] I. G. Macdonald, Symmetric functions and Hall polynomials, Oxford university press (1998).

[31] I. G. Macdonald, A new class of symmetric functions, Publ. IRMA Strasbourg, 372, (1988), 131-171.

[32] I. G. Macdonald, Affine Hecke algebras and orthogonal polynomials, (Vol. 157). Cambridge University Press, (2003).

[33] I. G. Macdonald, Orthogonal polynomials associated with root systems. In Orthogonal polynomials, Springer, Dordrecht,(1990), pp. 311-318.

[34] Y. Matsuno, Recurrence formula and conserved quantity of the Benjamin-Ono equation, Journal of the Physical Society of Japan, 52(9), (1983), 2955-2958.

[35] M. G. Matushko, Calogero-Sutherland system at free fermion point, to appear in Theoretical and Mathematical Physics

[36] A. Molev, M. Nazarov and G. Olshanski, Yangians and classical Lie algebras, Russian Math. Surveys **51** (1996), 205–282.

[37] I. Moser, Three integrable hamiltonian sysems connected with isospectrum deformations Adv. Math., 1976, 16, 354-370.

[38] M.Nazarov, On the spin Calogero-Sutherland model at infinity, Representations and Nilpotent Orbits of Lie Algebraic Systems. Birkhäuser, Cham, (2019), 421-439.

[39] M.L. Nazarov and E.K. Sklyanin, Integrable hierarchy of the quantum Benjamin-Ono equation Symmetry, Integrability and Geometry: Methods and Applications SIGMA **9** (2013), 078.

[40] M.L. Nazarov and E. K. Sklyanin, Sekiguchi-Debiard operators at infinity, Communications in Mathematical Physics **324**:3 (2013), 831-849.

[41] A. Okounkov, R. Pandharipande, Quantum cohomology of the Hilbert scheme of points in the plane, Inventiones mathematicae, 179(3), (2010), 523-557.

[42] M.A. Olshanetsky, A. M. Perelomov, Quantum integrable systems related to Lie algebras, Physics Reports, 94(6), (1983),313-404.
M.A. Olshanetsky, A. M. Perelomov, Classical integrable finite-dimensional systems related to Lie algebras, Physics Reports, 71(5), (1981), 313-400.

[43] H. Ono, Algebraic solitary waves in stratified fluids, J. Phys. Soc. Japan 39, (1975), 1082–1091.

[44] E.M. Opdam, Root systems and hypergeometric functions III, Compositio Mathematica, 67(1), (1988), 21-49.

[45] A.K. Pogrebkov, Boson-fermion correspondence and quantum integrable and dispersionless models, Russian Mathematical Surveys (2003) 58(5), 1003.

[46] A. P. Polychronakos, Exchange operator formalism for integrable systems of particles, Physical Review Letters **69**:5 (1992), 703.

[47] A. P. Polychronakos, Waves and solitons in the continuum limit of the Calogero-Sutherland model, Physical review letters, 74(26), (1995), 5153.

[48] N. Reshetikhin, Degenerate integrability of quantum spin Calogero–Moser systems, Letters in Mathematical Physics, 107(1), (2017), 187-200.

[49] N. Reshetikhin, Degenerate integrability of the spin Calogero–Moser systems and the duality with the spin Ruijsenaars systems, Letters in Mathematical Physics, 63(1), (2003), 55-71.

[50] P. Rossi, Gromov–Witten invariants of target curves via symplectic field theory, Journal of Geometry and Physics **58**:8 (2008), 931-941.

[51] J. Sekiguchi, Zonal spherical functions on some symmetric spaces, Publications of the Research Institute for Mathematical Sciences, 12(Supplement), (1977), 455-459.

[52] A. N. Sergeev, A. P. Veselov, Dunkl operators at infinity and Calogero-Moser systems, International Mathematics Research Notices, **21** (2015), 10959-10986.

[53] A.N. Sergeev, A.P. Veselov, Calogero-Moser operators in infinite dimension, eprint arXiv:0910.1984 (2009).

[54] R. P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge Univ. Press, Cambridge (1997).

[55] B. Sutherland, Exact results for a quantum many-body problem in one dimension, Physical Review A, 4(5), (1971), 2019.
B. Sutherland, Exact results for a quantum many-body problem in one dimension. II, Physical Review A 5.3 (1972): 1372.

[56] K. Takemura, D. Uglov, The orthogonal eigenbasis and norms of eigenvectors in the spin Calogero-Sutherland model, Journal of Physics A: Mathematical and General, 30(10), (1997), 3685.

[57] D. Uglov, Symmetric functions and the Yangian decomposition of the Fock and Basic modules of the affine Lie algebra $\hat{sl}(N)$, arXiv preprint q-alg/9705010 (1997).

[58] Uglov D. Yangian actions on higher level irreducible integrable modules of affine $\widehat{\mathfrak{gl}}_N$, eprint arXiv preprint math/9802048. – 1998.