Dynamics of quantum integrable models via coupled Heisenberg equations: models with Onsager algebra and more

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Exactly solvable models and algebras

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Montreal, Oct 07, 2022

Quantum dynamics of an observable

H – Hamiltonian

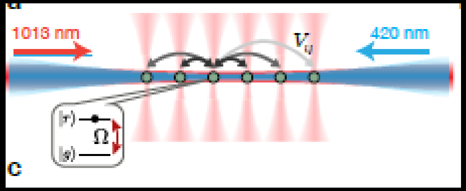
O – (Schrödinger operator of) observable

ρ_0 – initial state

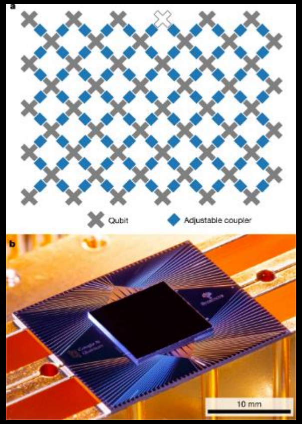
ρ_t – time-evolving state

von Neumann equation: $i\partial_t \rho_t = [H, \rho_t]$

evolution of the observable: $\langle O \rangle_t \equiv \operatorname{tr} \rho_t \, O$



Bernien et al
Nature 2017



Arute at al Nature 2019

Quantum dynamics in integrable models

To calculate $\langle O \rangle_t$ is usually a daunting task ***despite integrability***

- formfactor expansion: $$\langle O \rangle_t = \sum_{E,E'} \langle E|\mathrm{in}\rangle \langle \mathrm{in}|E'\rangle \langle E'|O|E\rangle \, e^{-i(E-E')t}$$
- quench action
- generalized hydrodynamics
- ...

Heisenberg representation

$$i\partial_t \rho_t = [H, \rho_t] \qquad \Leftrightarrow \qquad \rho_t = e^{-iHt} \rho_0 \, e^{iHt}$$

$$\langle O \rangle_t = \mathrm{tr}\Big(e^{-iHt} \rho_0 \, e^{iHt} \, O \Big) = \mathrm{tr}\, \rho_0 \, O_t$$

Heisenberg operator

$$O_t = e^{iHt} O \, e^{-iHt}$$

Heisenberg operator

Schrödinger operator

Heisenberg equation: $\partial_t O_t = i[H, O_t]$

Heisenberg equation

$$O_t = e^{iHt} O\, e^{-iHt} \qquad\qquad \langle O \rangle_t = \mathrm{tr}\, \rho_0\, O_t$$

Heisenberg equation: $\partial_t O_t = i[H, O_t], \qquad O_0 = O$

technically, proceed as $[H, O_t] = e^{iHt}\, [H, O]\, e^{-iHt} \equiv [H, O]_t$

$$\partial_t(...) = i[H, ...]$$

Heisenberg operators are also useful
for calculating correlation functions: $\langle O_t\, \tilde{O} \rangle_\beta \equiv \mathrm{tr}\big(O_t\, \tilde{O}\, e^{-\beta H}\big)/\mathrm{tr}\, e^{-\beta H}$

Transverse-field Ising model: Hamiltonian

$$H = a_1 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x - a_0 \sum_{j=1}^{N} \sigma_j^z$$

notations for translation-invariant operators:

$$\sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x \to \sigma^x \sigma^x$$

$$\sum_{j=1}^{N} \sigma_j^x \sigma_{j+2}^y \to \sigma^x \, \mathbb{1} \, \sigma^y$$

$$H = a_1 \sigma^x \sigma^x - a_0 \sigma^z$$

Transverse-field Ising model: Heisenberg equations

observable: $A^1 = \sigma^x \sigma^x$ Hamiltonian: $H = a_1 \sigma^x \sigma^x - a_0 \sigma^z$

$$i[H, A^1] = 2a_0(\sigma^x \sigma^y + \sigma^y \sigma^x)$$

$$i[H, \sigma^x \sigma^y + \sigma^y \sigma^x] : \qquad \sigma^z, \quad \sigma^x \sigma^x, \quad \sigma^y \sigma^y, \quad \sigma^x \sigma^z \sigma^x$$

$$i[H, \sigma^z] : \qquad \sigma^x \sigma^y + \sigma^y \sigma^x$$

$$i[H, \sigma^y \sigma^y] : \qquad \sigma^x \sigma^y + \sigma^y \sigma^x, \quad \sigma^x \sigma^z \sigma^y + \sigma^y \sigma^z \sigma^x$$

$$i[H, \sigma^x \sigma^z \sigma^x] : \qquad \sigma^x \sigma^y + \sigma^y \sigma^x, \quad \sigma^x \sigma^z \sigma^y + \sigma^y \sigma^z \sigma^x$$

$$i[H, \sigma^x \sigma^z \sigma^y + \sigma^y \sigma^z \sigma^x] : \qquad \sigma^x \sigma^z \sigma^z \sigma^x, \quad \sigma^x \sigma^z \sigma^x, \quad \sigma^y \sigma^z \sigma^y, \quad \sigma^y \sigma^y$$

Transverse-field Ising model: Heisenberg equations

$$\partial_t G_t^n = 2\,i\left(a_0(-A_t^n + A_t^{-n}) + a_1(-A_t^{1+n} + A_t^{1-n})\right)$$

$$\partial_t A_t^n = -4\,i\left(a_0\, G_t^n + a_1\, G_t^{n-1}\right) \qquad n \in \mathbb{Z}$$

$$G^n = (i/2)\big(\sigma^x \underbrace{\sigma^z\sigma^z...\sigma^z}_{n-1}\sigma^y + \sigma^y \underbrace{\sigma^z\sigma^z...\sigma^z}_{n-1}\sigma^x\big)$$

$$A^n = \sigma^x \underbrace{\sigma^z\sigma^z...\sigma^z}_{n-1}\sigma^x$$

$$G^{-n} = -G^n$$

$$A^{-n} = \sigma^y \underbrace{\sigma^z\sigma^z...\sigma^z}_{n-1}\sigma^y$$

$$G^0 = 0$$

Onsager algebra

$$A^0 = -\sigma^z$$

$$n \geq 0$$

Solving Heisenberg equations

$$\partial_t G_t^n = 2\,i\left(a_0(-A_t^n + A_t^{-n}) + a_1(-A_t^{1+n} + A_t^{1-n})\right)$$

$$\partial_t A_t^n = -4\,i\left(a_0\,G_t^n + a_1\,G_t^{n-1}\right) \qquad n \in \mathbb{Z}$$

$$\partial_t^2 G_t^n = -16\left(a_0\,a_1\,G_t^{n-1} + (a_0^2 + a_1^2)G_t^n + a_0\,a_1\,G_t^{n+1}\right), \qquad n = 1, 2, \ldots$$

$$\boxed{\begin{aligned} G_t^n &= \textstyle\sum_{m=1}^{\infty}\left(\partial_t c_t^{nm}\,G^m + 2\,i\,c_t^{nm}\left(a_0(-A^m + A^{-m}) + a_1(-A^{1+m} + A^{1-m})\right)\right) \\ c_t^{nm} &\equiv (2/\pi)\int_0^\pi d\varphi\,\sin(n\varphi)\sin(m\varphi)\sin(\varepsilon_\varphi t)\varepsilon_\varphi^{-1} \qquad \varepsilon_\varphi \equiv 4\sqrt{a_0^2 + a_1^2 + 2a_0\,a_1\cos\varphi} \end{aligned}}$$

$$A_t^n = \int_0^t \partial_{t'} A_{t'}^n = \ldots$$ (explicit but bulky expression)

Dynamics after a quench

$$G_t^n = \sum_{m=1}^{\infty} \Big(\partial_t c_t^{nm}\, G^m + 2\,i\,c_t^{nm}\big(a_0(-A^m + A^{-m}) + a_1(-A^{1+m} + A^{1-m})\big)\Big)$$

$$\langle O \rangle_t = \mathrm{tr}\,\rho_0\, O_t \equiv \langle O_t \rangle$$

$$\rho_0 = \bigotimes_{m=1}^{N} \left(\frac{1}{2}(1 + \mathbf{p}\boldsymbol{\sigma})\right), \quad \mathbf{p} = (p_x, p_y, p_z), \quad |\mathbf{p}| \le 1, \quad \boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$$

$$\langle A^n \rangle = N p_x^2\, p_z^{n-1}, \quad \langle A^{-n} \rangle = N p_y^2\, p_z^{n-1}, \quad \langle A^0 \rangle = -N p_z, \quad n = 1, 2, \ldots$$

Dynamics after a quench

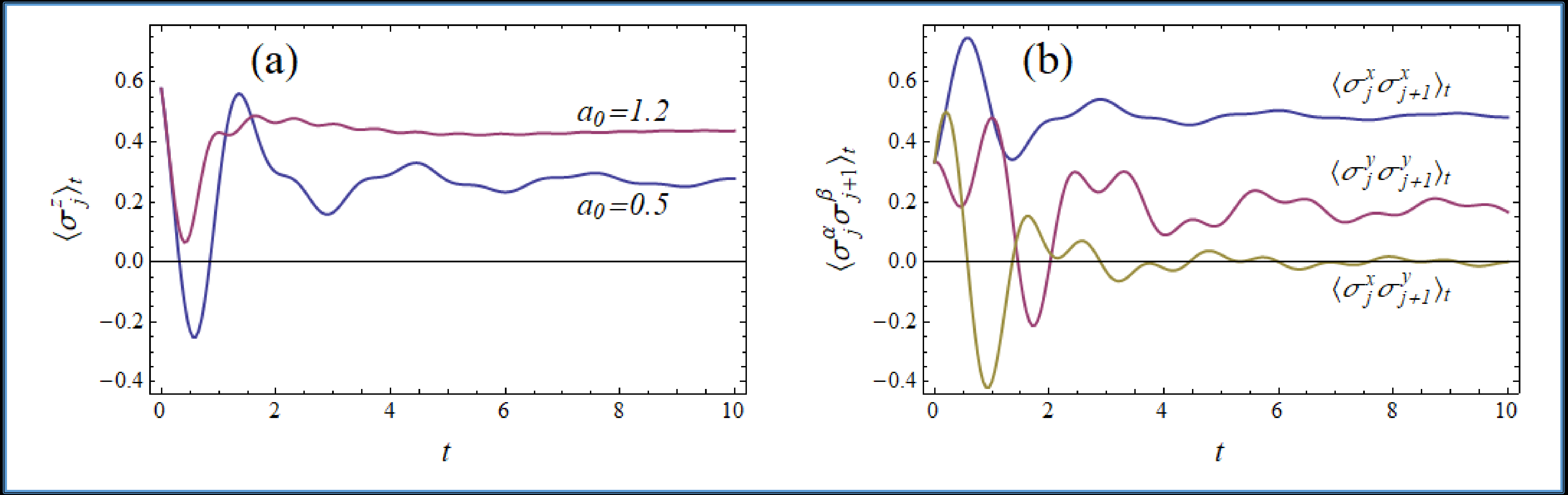
$$\begin{aligned}\langle A^n\rangle_t =&\langle A^n\rangle_0 + 4N\int_0^\pi \frac{d\varphi}{\pi}\Big(a_0\sin\left(n\varphi\right) + a_1\sin\left((n-1)\varphi\right)\Big)\,\sin\varphi\\ &\times\left(R_\varphi\frac{\sin\varepsilon_\varphi t}{\varepsilon_\varphi} + Q_\varphi\frac{1-\cos\varepsilon_\varphi t}{\varepsilon_\varphi^2}\right)\end{aligned}$$

$$\langle G^n\rangle_t = iN\int_0^\pi \frac{d\varphi}{\pi}\sin(n\varphi)\,\sin\varphi\left(R_\varphi\cos\varepsilon_\varphi t + Q_\varphi\frac{\sin\varepsilon_\varphi t}{\varepsilon_\varphi}\right)$$

$$R_\varphi = \frac{2\,p_x\,p_y}{1+p_z^2-2p_z\,\cos\varphi}$$

$$Q_\varphi = -4\,a_1\left(\frac{p_x^2\,p_z - p_y^2/p_z + (a_0/a_1)(p_x^2-p_y^2)}{1+p_z^2-2p_z\cos\varphi} + p_y^2/p_z + p_z\right)$$

Dynamics after a quench



$$\mathbf{p} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$$

Transverse-field Ising model: noninteracting fermions

Jordan-Wigner transformation:

$$c_j^\dagger \equiv \sigma_j^+ \Pi_{j-1}, \quad c_j \equiv \sigma_j^- \Pi_{j-1}, \quad \{c_j^\dagger, c_l\} = \delta_{jl}, \quad \{c_j, c_l\} = 0$$

$$\Pi_n \equiv \prod_{j=1}^{n} \sigma_j^z, \quad \sigma_j^+ = (\sigma_j^x + i\sigma_j^y)/2, \quad \sigma_j^- = (\sigma_j^x - i\sigma_j^y)/2, \quad |\mathrm{vac}\rangle = |\downarrow\downarrow \ldots \downarrow\rangle$$

$$H = \sum_j (c_j^\dagger c_{j-1} + c\, c_{j-1} + h.c.) - a_0 \sum_j c_j^\dagger c_j$$

$A^n,\, G^n$ are quadratic forms in $c_j^\dagger\, c_j$, e.g.

$$A^n \sim \sum_j (c_j + c_j^\dagger)(c_{j+n-1} + c_{j+n-1}^\dagger), \quad n \geq 1$$

Transverse-field Ising model: noninteracting fermions

in the fermionic representation an initial state can be hard to handle:

$$\rho_0 = \bigotimes_{m=1}^{N} \left(\frac{1}{2}(1 + \mathbf{p}\boldsymbol{\sigma}) \right)$$

$$p_x = \cos\phi \, \sin\theta, \quad p_y = \sin\phi \, \sin\theta, \quad p_z = \cos\theta$$

$$|\mathrm{in}\rangle = e^{i(\phi/2)\sum_j \sigma_j^z} e^{i(\theta/2)\sum_j \sigma_j^x} |\downarrow\downarrow \ldots \downarrow\rangle, \qquad \rho_0 = |\mathrm{in}\rangle\langle\mathrm{in}|$$

superposition of states with all possible fermion numbers

Onsager algebra

structure:
$$[A^n, A^m] = 4\,G^{n-m},$$
$$[G^n, A^m] = 2A^{m+n} - 2A^{m-n},$$
$$[G^n, G^m] = 0$$

can be generated from $A^0,\ A^1$ recursively through

$$G^n = \frac{1}{4}[A^n, A^0], \qquad n = 0, 1, 2, \ldots,$$
$$A^{n+1} - A^{n-1} = \frac{1}{2}[G^1, A^n], \qquad n = 0, \pm 1, \pm 2, \ldots \qquad (1)$$

if and only if the Dolan-Grady conditions (1982) are satisfied:

$$[A_0, [A_0, [A_0, A_1]]] = 16[A_0, A_1],$$
$$[A_1, [A_1, [A_1, A_0]]] = 16[A_1, A_0].$$

Onsager algebra in nearest-neighbor spin models

Gehlen and Rittenberg (1985):

for each $n \geq 1$ a spin-$n/2$ representation exists;

the corresponding Hamiltonian

$$H = a_0\, A^0 + a_1\, A^1$$

is a nearest-neighbor Hamiltonian:

spin $1/2$ - transverse-field Ising model

higher spins – superintegrable chiral n-state Potts models

in contrast to Ising model, can not be mapped to free fermions ???

see, however, works by Minami (2016-2021) on generalized Jordan-Wigner transformation

Dynamics in models with Onsager algebra

$$H = a_0\,A^0 + a_1\,A^1$$

Heisenberg equations and their solution are independent on the representation

$$G_t^n = \sum_{m=1}^{\infty}\left(\partial_t c_t^{nm}\,G^m + 2\,i\,c_t^{nm}\big(a_0(-A^m + A^{-m}) + a_1(-A^{1+m} + A^{1-m})\big)\right)$$

$$c_t^{nm} \equiv (2/\pi)\int_0^{\pi} d\varphi\,\sin(n\varphi)\sin(m\varphi)\sin(\varepsilon_\varphi t)\varepsilon_\varphi^{-1} \qquad \varepsilon_\varphi \equiv 4\sqrt{a_0^2 + a_1^2 + 2a_0\,a_1\cos\varphi}$$

3-state superintegrable chiral Potts model

$$A^0 = \frac{4}{3}\sum_{j=1}^{N}\left(\frac{\tau_j}{1-\omega^*} + h.c.\right) = \frac{4}{3}\sum_{j=1}^{N} S_j^z,$$

$$A^1 = \frac{4}{3}\sum_{j=1}^{N}\left(\frac{\sigma_j\sigma_{j+1}^\dagger}{1-\omega^*} + h.c.\right), \qquad \omega = e^{2\pi i/3}$$

$$\tau_j^3 = \mathbb{1}_j, \qquad \sigma_j^3 = \mathbb{1}_j, \qquad \tau_j^2 = \tau_j^\dagger, \qquad \sigma_j^2 = \sigma_j^\dagger, \qquad \sigma_j\tau_j = \omega\,\tau_j\sigma_j$$

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^* \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

3-state Potts model

$$G_t^n = \sum_{m=1}^{\infty} \left(\partial_t c_t^{nm}\, G^m + 2\, i\, c_t^{nm} \left(a_0(-A^m + A^{-m}) + a_1(-A^{1+m} + A^{1-m}) \right) \right)$$

in contrast to Ising model, general formulae for $A^n,\ G^n$ are unknown

we generate them recursively one by one using computer algebra code

$$G^1 = \frac{4}{9} \frac{1}{1-\omega^*} \sum_{j=1}^{N} \sigma_j (\tau_j + \tau_j^\dagger - \tau_{j+1} - \tau_{j+1}^\dagger) \sigma_{j+1}^\dagger \ - h.c.$$

for higher *n* expressions rapidly become bulky

n	1	2	3	4
number of terms in A^n	1	9	43	181

3-state Potts model

In general, no apparent structure is seen in $A^n,\ G^n$

certain properties can be guessed, however

complete set of single-site operators:

$$\{\mathbb{1}, \tau, \tau^\dagger, \sigma, \sigma^\dagger, \sigma\tau, \tau^\dagger\sigma^\dagger, \sigma^\dagger\tau, \tau^\dagger\sigma\}$$

non-shifting operators: $\{\mathbb{1}, \tau, \tau^\dagger\}$

shifting operators: $\{\sigma, \sigma^\dagger, \sigma\tau, \tau^\dagger\sigma^\dagger, \sigma^\dagger\tau, \tau^\dagger\sigma\}$

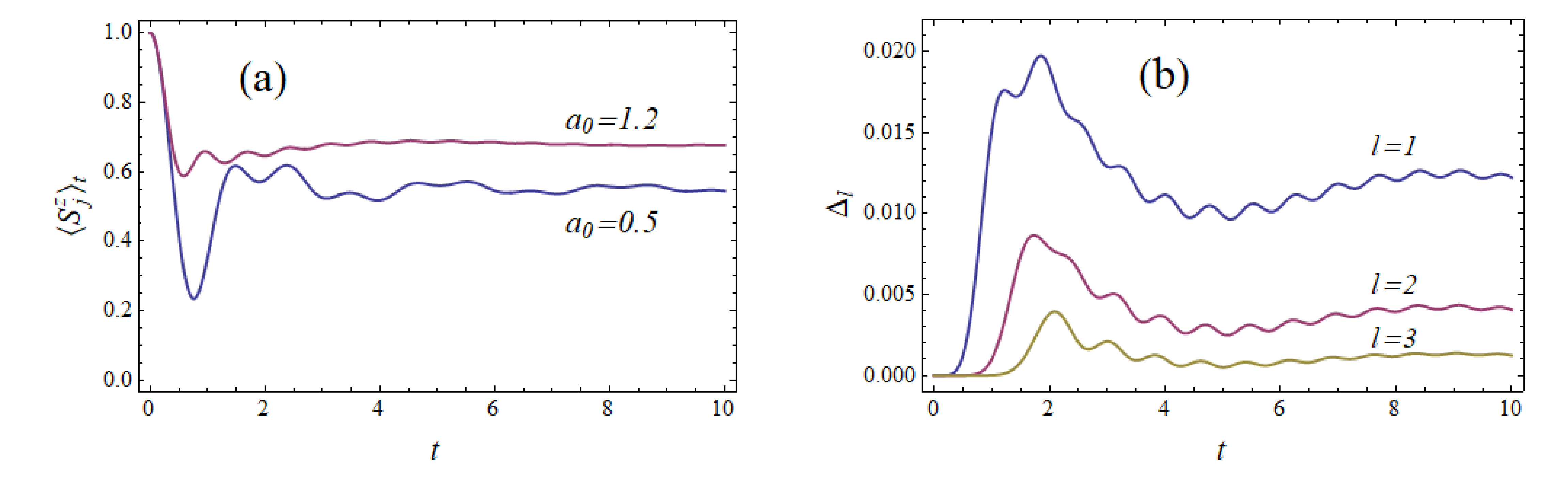
3-state Potts model

conjectured properties of A^n, G^n :

- each G^n, A^n is a sum of *strings*, where a string of length m is a tensor product of m non-identity single-site operators acting on m consecutive sites
- each string of length $n \geq 2$ has shifting operators at both its ends
- each G^n consists of strings of lengths not less than 2.
- each A^n with odd n consists of strings of lengths not less than 2
- each A^n with even n contains strings of length 1 consisting of operators τ_j, $\tau_j^\dagger$

example of a term entering A^4 : $\sigma_j^\dagger\,(\tau_{j+1}^\dagger\sigma_{j+1}^\dagger)\,(\tau_{j+2}^\dagger\sigma_{j+2})\,(\tau_{j+3}^\dagger\sigma_{j+3}^\dagger)\sigma_{j+4}^\dagger$

3-state Potts model: dynamics after a quench



$$\mathbf{p} = (0, 0, 1)$$

Open problems - I

- explicit formulae for n-state representations of Onsager algebra
  (or, at least, proof of their properties)
- site-resolved dynamics
    - straightforward for Ising model (work in progress with Igor Ermakov)
    - higher spins?
    - local analog of Onsager algebra?

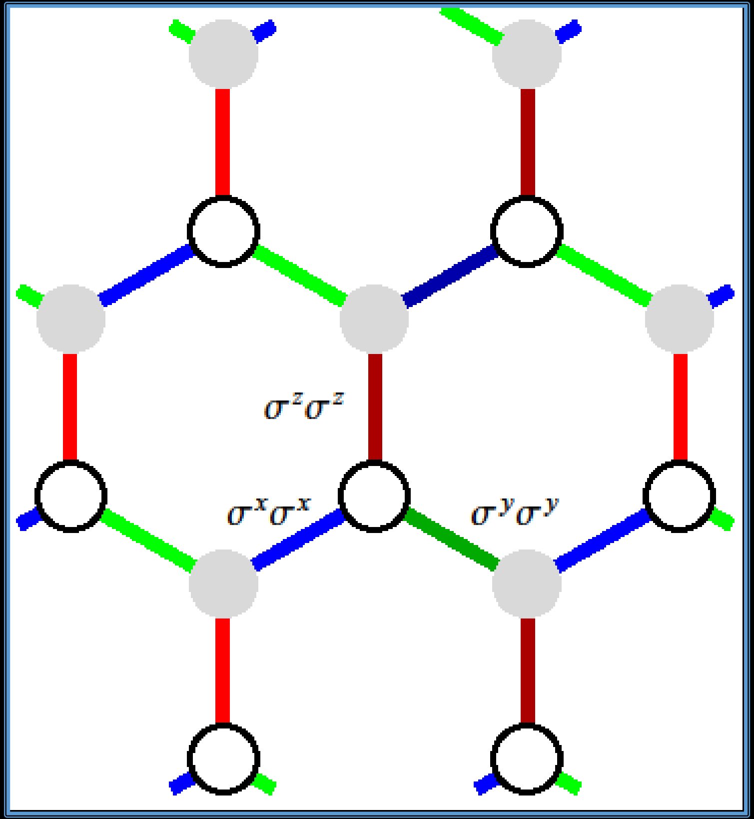
Open problems - II

- models with non-local representations of Onsager algebra?
- time-dependent Hamiltonians?
- integrable open systems?

Kitaev model on a honeycomb lattice

$$H = J_x \sum_{x-\text{links}} \sigma^x_{\boldsymbol{j}A}\sigma^x_{\boldsymbol{j}'B} + J_y \sum_{y-\text{links}} \sigma^y_{\boldsymbol{j}A}\sigma^y_{\boldsymbol{j}'B} + J_z \sum_{z-\text{links}} \sigma^z_{\boldsymbol{j}A}\sigma^z_{\boldsymbol{j}'B}$$

we consider $J_x = J_y = J_z = 1$



mapping to Majorana fermions:

$$H = \sum_{\text{links}} I_{\boldsymbol{j}\boldsymbol{j}'}\, c_{\boldsymbol{j}A} c_{\boldsymbol{j}'B}$$

local integrals of motion

quadratic model when IoMs fixed

sum over disorder configurations for generic states!

Kitaev model on a Bethe lattice: routs and strings

route: sequence of turns, left or right, e.g.

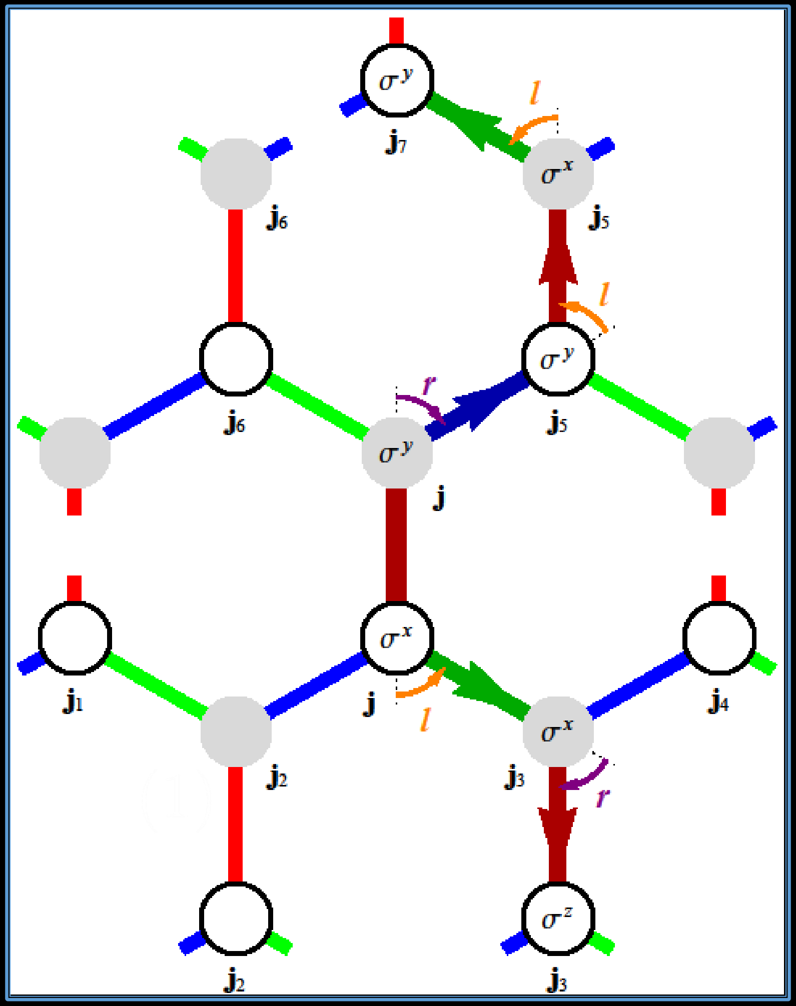
$$\mathscr{V} = rll, \qquad \mathscr{W} = lr.$$

string: operator constructed from a link and two routes:

$$\underset{j}{Z}{}^{rll}_{lr} = \sigma^y_{j_7 A}\,\sigma^x_{j_5 B}\,\sigma^y_{j_5 A}\,\sigma^y_{j B}\,\sigma^x_{j A}\,\sigma^x_{j_3 B}\,\sigma^z_{j_3 A}$$

the same operator has different string representations:

$$\underset{j_3}{Z}{}^{lrrll}_{\emptyset}, \quad \underset{j}{Y}{}^{r}_{rrll}, \quad \underset{j}{Z}{}^{rll}_{lr}, \quad \underset{j_5}{X}{}^{llr}_{ll}, \quad \underset{j_5}{Z}{}^{l}_{rllr}, \quad \underset{j_7}{Y}{}^{rrllr}_{\emptyset}$$

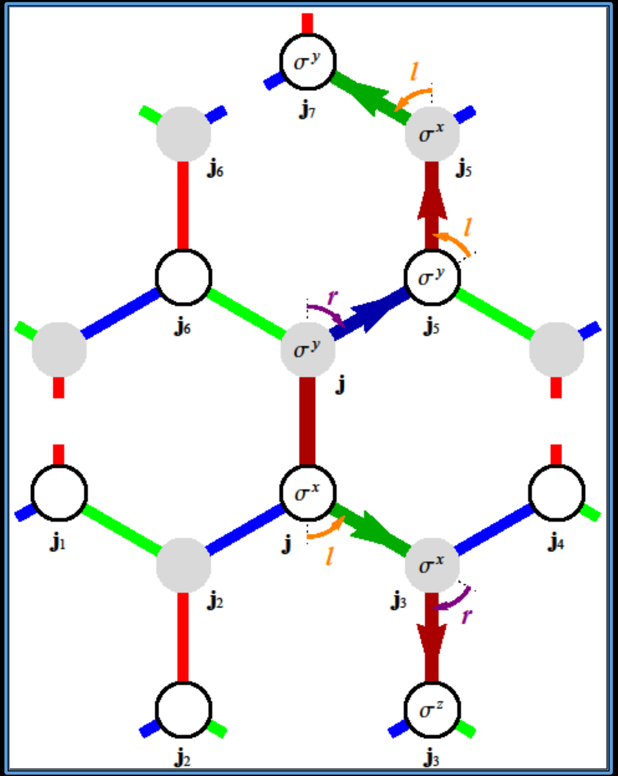


Time derivative of strings

strings form an algebra wrt commutation

$$\left[iH, \underset{\mathbf{j}}{Q^{\mathscr{V}}_{\mathscr{W}}}\right] = \mathrm{Ex}\left[\underset{\mathbf{j}}{Q^{\mathscr{V}}_{\mathscr{W}}}\right] + 2\,\frac{\mathrm{sign}\left(\mathscr{V}^{\smile}\right)}{\mathrm{sign}\left(\mathscr{V}\right)}\,\underset{\mathbf{j}}{Q^{\mathscr{V}^{\smile}}_{\mathscr{W}}} + 2\,\frac{\mathrm{sign}\left(\mathscr{W}^{\smile}\right)}{\mathrm{sign}\left(\mathscr{W}\right)}\,\underset{\mathbf{j}}{Q^{\mathscr{V}}_{\mathscr{W}^{\smile}}}, \qquad |\mathscr{V}|, |\mathscr{W}| \geq 1$$

$$\left[iH, \underset{\mathbf{j}}{Q^{\emptyset}_{\emptyset}}\right] = \mathrm{Ex}\left[\underset{\mathbf{j}}{Q^{\emptyset}_{\emptyset}}\right]$$



$$Q = X, Y, Z$$

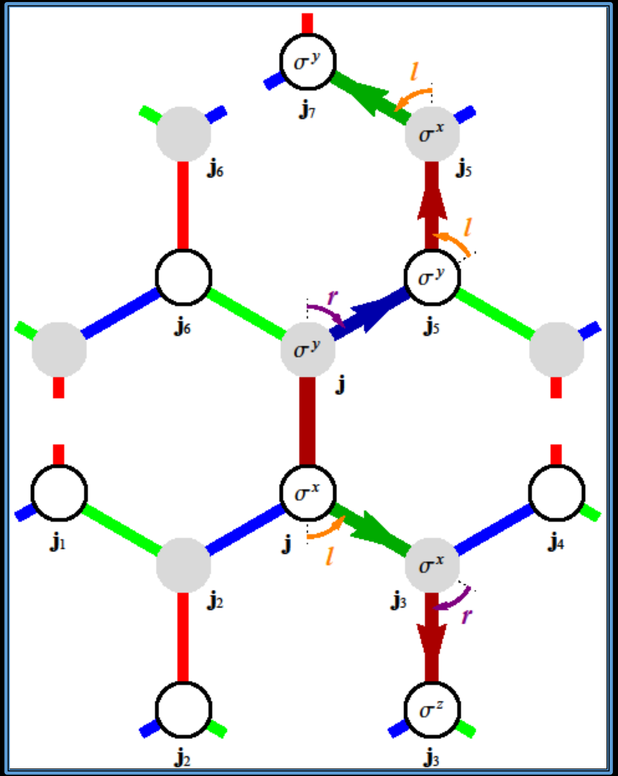
$$\mathrm{Ex}\left[\underset{\mathbf{j}}{Q^{\mathscr{V}}_{\mathscr{W}}}\right] = 2\left(-\underset{\mathbf{j}}{Q^{\mathscr{V}r}_{\mathscr{W}}} + \underset{\mathbf{j}}{Q^{\mathscr{V}l}_{\mathscr{W}}} - \underset{\mathbf{j}}{Q^{\mathscr{V}}_{\mathscr{W}r}} + \underset{\mathbf{j}}{Q^{\mathscr{V}}_{\mathscr{W}l}}\right)$$

Time derivative of strings

$$[iH, \underset{\mathbf{j}}{X^{\mathscr{V}}_{\emptyset}}] = \mathrm{Ex}\left[\underset{\mathbf{j}}{X^{\mathscr{V}}_{\emptyset}}\right] + 2\,\frac{\mathrm{sign}\left(\mathscr{V}\breve{\;}\right)}{\mathrm{sign}\left(\mathscr{V}\right)}\,\underset{\mathbf{j}}{X^{\mathscr{V}\breve{\;}}_{\emptyset}} - 2\,\frac{\mathrm{sign}\left(\breve{\;}\mathscr{V}\right)}{\mathrm{sign}\left(\mathscr{V}\right)} \times \begin{cases} \underset{\mathbf{j}_1}{Y^{\emptyset}_{\breve{\mathscr{V}}}}, & \mathscr{V}_1 = r \\ \underset{\mathbf{j}_2}{Z^{\emptyset}_{\breve{\mathscr{V}}}}, & \mathscr{V}_1 = l \end{cases}$$

$$[iH, \underset{\mathbf{j}}{Y^{\mathscr{V}}_{\emptyset}}] = \mathrm{Ex}\left[\underset{\mathbf{j}}{Y^{\mathscr{V}}_{\emptyset}}\right] + 2\,\frac{\mathrm{sign}\left(\mathscr{V}\breve{\;}\right)}{\mathrm{sign}\left(\mathscr{V}\right)}\,\underset{\mathbf{j}}{Y^{\mathscr{V}\breve{\;}}_{\emptyset}} - 2\,\frac{\mathrm{sign}\left(\breve{\;}\mathscr{V}\right)}{\mathrm{sign}\left(\mathscr{V}\right)} \times \begin{cases} \underset{\mathbf{j}_3}{Z^{\emptyset}_{\breve{\mathscr{V}}}}, & \mathscr{V}_1 = r \\ \underset{\mathbf{j}_4}{X^{\emptyset}_{\breve{\mathscr{V}}}}, & \mathscr{V}_1 = l \end{cases}$$

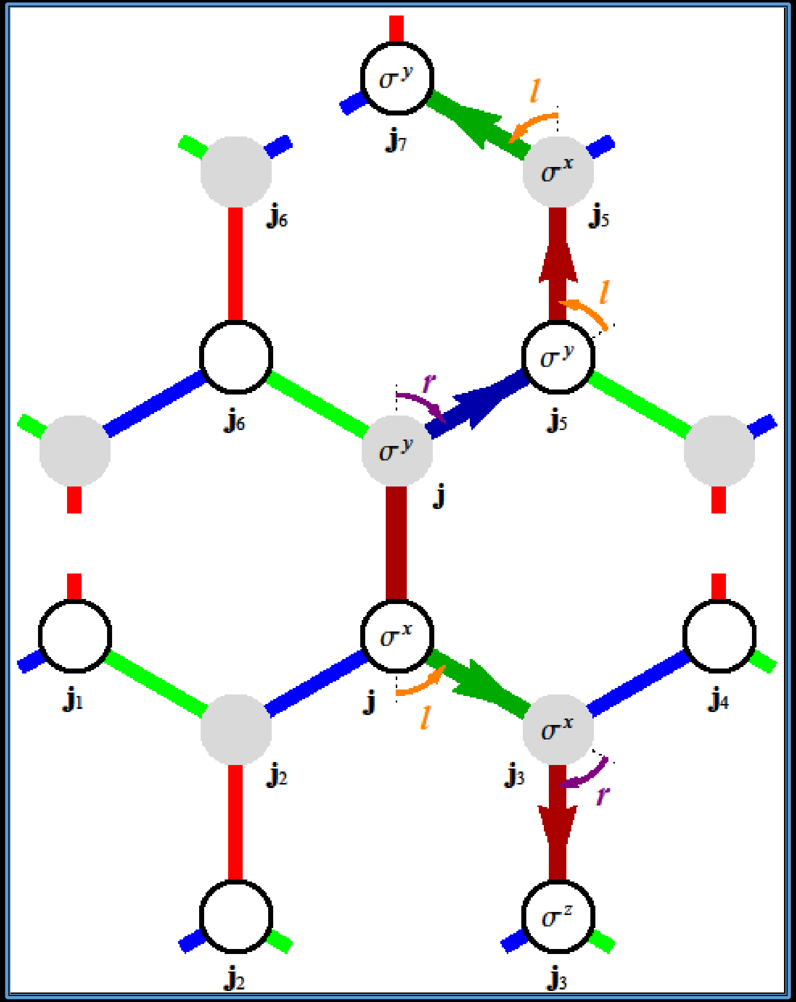
$$[iH, \underset{\mathbf{j}}{Z^{\mathscr{V}}_{\emptyset}}] = \mathrm{Ex}\left[\underset{\mathbf{j}}{Z^{\mathscr{V}}_{\emptyset}}\right] + 2\,\frac{\mathrm{sign}\left(\mathscr{V}\breve{\;}\right)}{\mathrm{sign}\left(\mathscr{V}\right)}\,\underset{\mathbf{j}}{Z^{\mathscr{V}\breve{\;}}_{\emptyset}} - 2\,\frac{\mathrm{sign}\left(\breve{\;}\mathscr{V}\right)}{\mathrm{sign}\left(\mathscr{V}\right)} \times \begin{cases} \underset{\mathbf{j}_5}{X^{\emptyset}_{\breve{\mathscr{V}}}}, & \mathscr{V}_1 = r \\ \underset{\mathbf{j}_6}{Y^{\emptyset}_{\breve{\mathscr{V}}}}, & \mathscr{V}_1 = l \end{cases}$$



Sum over sites

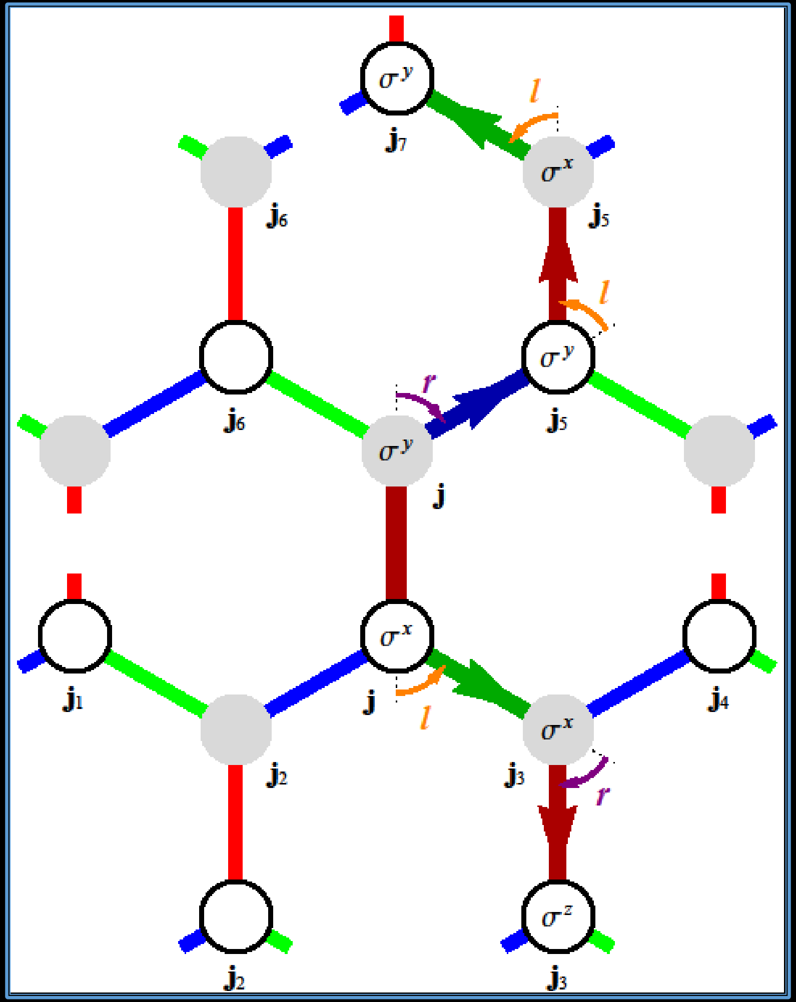
$$Q^{\mathscr{V}}_{\mathscr{W}} = \sum_{\mathbf{j}}{}' \, Q^{\mathscr{V}}_{\mathscr{W}\,\mathbf{j}}$$

removes subscripts **j** from all equations



Sum over strings of equal length

$$Q^{m\,n} = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^{n+m} \sum_{\substack{\mathscr{V},\,\mathscr{W}:\\ |\mathscr{V}|=m\\ |\mathscr{W}|=n}} \operatorname{sign}\mathscr{V}\,\operatorname{sign}\mathscr{W}\;\left(Q^{\mathscr{V}}_{\mathscr{W}} + Q^{\mathscr{W}}_{\mathscr{V}}\right)$$



reduces the size of algebra from exponential to polynomial

Heisenberg equations

$$\mathcal{X}_t^{m\,n} \equiv X_t^{m\,n} - \frac{1}{2}\left(Y_t^{m\,n} + Z_t^{m\,n}\right)$$

$$\mathcal{Y}_t^{m\,n} \equiv Y_t^{m\,n} - \frac{1}{2}\left(Z_t^{m\,n} + X_t^{m\,n}\right)$$

$$\mathcal{Z}_t^{m\,n} \equiv Z_t^{m\,n} - \frac{1}{2}\left(X_t^{m\,n} + Y_t^{m\,n}\right)$$

$$\partial_t\,\mathcal{Q}_t^{m\,n} = -2\sqrt{2}\left(\mathcal{Q}_t^{(m+1)\,n} + \mathcal{Q}_t^{m\,(n+1)} - \mathcal{Q}_t^{(m-1)\,n} - \mathcal{Q}_t^{m\,(n-1)}\right), \quad m,n \geq 1$$

$$\partial_t\,\mathcal{Q}_t^{0\,n} = -2\sqrt{2}\left(\mathcal{Q}_t^{1\,n} + \mathcal{Q}_t^{0\,(n+1)} - \frac{3}{2}\,\mathcal{Q}_t^{0\,(n-1)}\right), \qquad n \geq 1$$

$$\partial_t\,\mathcal{Q}_t^{0\,0} = -2\sqrt{2}\left(\mathcal{Q}_t^{1\,0} + \mathcal{Q}_t^{0\,1}\right) = 4\sqrt{2}\,\mathcal{Q}_t^{0\,1}$$

$$\mathcal{Q} = \mathcal{X}, \mathcal{Y}, \mathcal{Z}$$

Solving Heisenberg equations

$$\mathcal{Q}_t^{m\,n} = \sum_{0\le\tilde m\le\tilde n} \mathbb{G}^{m\,n}_{\tilde m\,\tilde n}(t)\,\mathcal{Q}^{\tilde m\,\tilde n}, \quad m \le n$$

$$\mathbb{G}^{mn}_{\tilde m\tilde n}(t) = \int_0^\pi\int_0^\pi \frac{dp}{\pi}\frac{dq}{\pi}\, e^{-iE(p,q)\,t}\chi_{\tilde m\tilde n}(p,q)\,\xi^{m\,n}(p,q)$$

$$\mathbb{G}^{m\,n}_{\tilde m\,\tilde n}(0) = \delta^{m\,n}_{\tilde m\,\tilde n}, \quad 0\le m\le n, \quad 0\le\tilde m\le\tilde n$$

$$E(p,q) = 4\sqrt{2}\,(\cos p + \cos q)$$

$$\xi^{m\,n}(p,q) = e^{\frac{i\pi}{2}(m+n)}\bigg(\Big(\sin(mp)\sin(nq) - 2\sin\big((m+1)p\big)\sin\big((n+1)q\big)\Big) + \{m\leftrightarrow n\}\bigg)$$

$$\chi_{\tilde m\,\tilde n}(p,q) = -(2-\delta_{\tilde m\,\tilde n})\,e^{-\frac{i\pi}{2}(\tilde m+\tilde n)}\sum_{l=1}^{\infty}\frac{1}{2^l}\Big(\sin\big((\tilde m+l)p\big)\sin\big((\tilde n+l)q\big) + \{\tilde m\leftrightarrow\tilde n\}\Big)$$

Dynamics after a quench

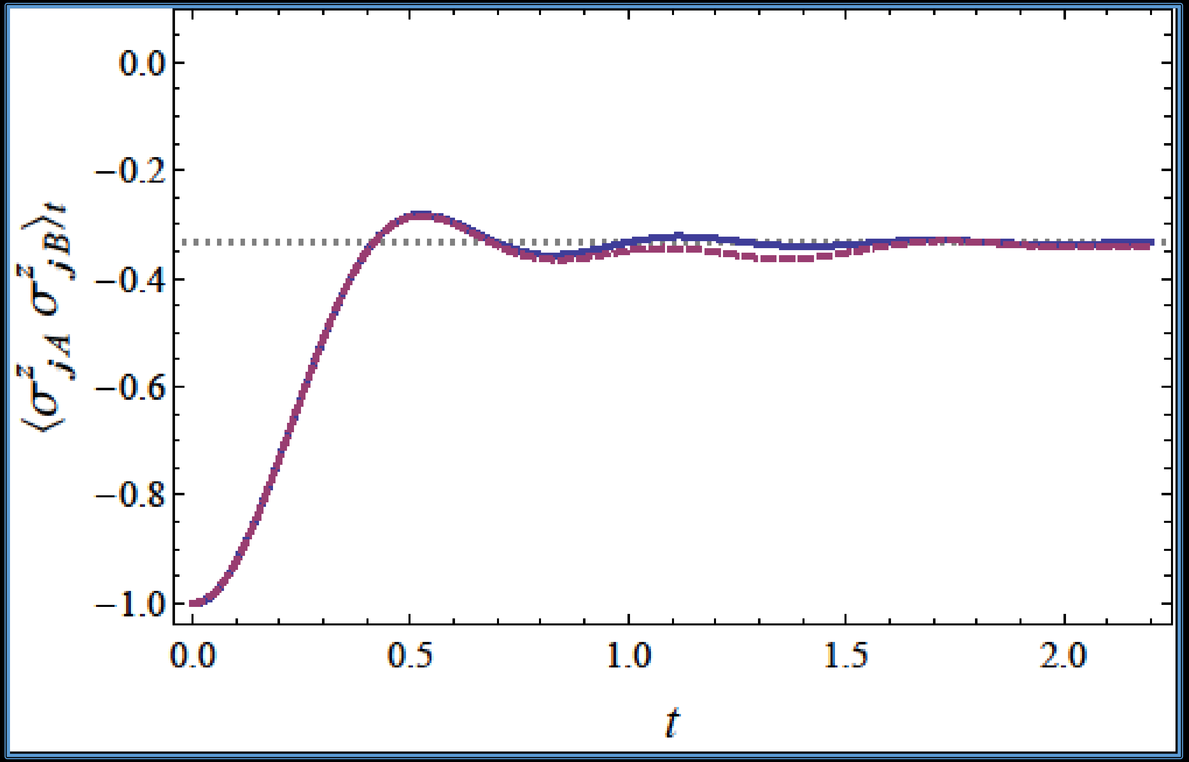
(staggered) translation-invariant product state:

$$\rho_0 = \bigotimes_{\mathbf{j}} \left(\frac{1}{2}(1 + \mathbf{p}\,\boldsymbol{\sigma}_{\mathbf{j}A}) \right) \left(\frac{1}{2}(1 + \eta\,\mathbf{p}\,\boldsymbol{\sigma}_{\mathbf{j}B}) \right), \quad \eta = \pm 1$$

$$\langle \sigma^z_{\mathbf{j}A}\, \sigma^z_{\mathbf{j}B} \rangle_t = \frac{2}{3}\eta \int_0^\pi \int_0^\pi \frac{dp}{\pi}\frac{dq}{\pi}\, e^{-iE(p,q)\,t} \chi_{00}(p,q)\,\xi^{00}(p,q)\left(p_z^2 - \frac{1}{2}(p_x^2 + p_y^2)\right) + \frac{1}{3}\eta\,(p_x^2 + p_y^2 + p_z^2),$$

Ktaev model: dynamics after a quench

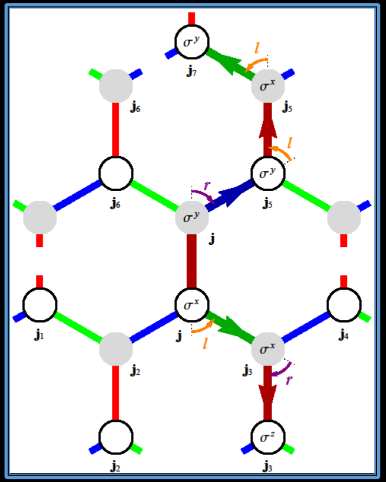
blue solid line – our result for Bethe lattice, magenta dashed line – numerical calculation for honeycomb lattice from [L. Rademaker, SciPost Phys. 7, 071 (2019)]



$$\mathbf{p} = (0, 0, 1), \quad \eta = -1$$

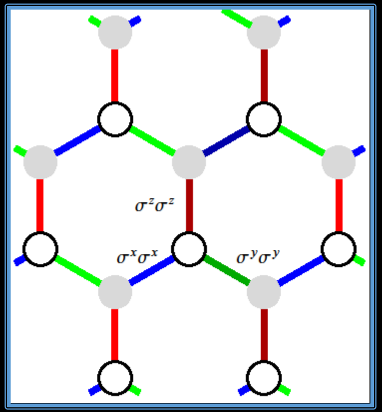
Why not true honeycomb lattice?

closed strings do not enter Heisenberg equations



strings that are “almost closed” (one edge missing) should be treated separately

even merely counting the fraction of closed strings among strings of a given length is an outstanding problem
(H. Duminil-Copin and S. Smirnov, 2012)



Open problems

- generalize to $J_x \neq J_y \neq J_z$ (feasible but tedious)
- site-resolved dynamics (feasible but tedious)
- generalize to true Honeycomb lattice (challenging :)

Further prospects

Are there any other models that can be addressed by the method?

Note: an algebra is redundant,

closeness wrt commutation *with the Hamiltonian alone* suffice

References:

O. Lychkovskiy, *Closed hierarchy of Heisenberg equations in integrable models with Onsager algebra*,
SciPost Phys. 10, 124 (2021)

O. Gamayun, O. Lychkovskiy, *Out-of-equilibrium dynamics of the Kitaev model on the Bethe lattice via a set of Heisenberg equations*,
SciPost Phys. 12, 175 (2022)

Thank you for your attention!